C. 10.

Motion of Projectile

Since the path of a projectile is parabolic, therefore it seems necessary to revise some particulars about parabola first and then to discuss the motion of projectile.

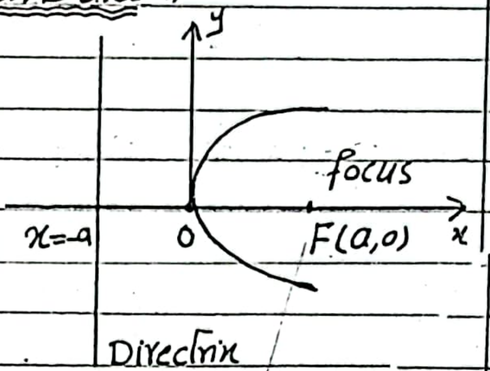
Standard Forms of Parbola

(a)# $y^2 = 4ax$

(1) Focus is at $F(a,0)$

(2) Vertex is at $O(0,0)$

(3) x-axis is the (equation) axis of parabola and its equation is $y = 0$



(4) Length of latus-Rectum $= 4a$

(5) $x = -a$ is the equation of directrix

Latus Rectum# The chord of a parabola which passes through the focus and is perpendicular to the axis of the parabola is called the latus-rectum of the parabola. OR Perpendicular focal chord of a parabola is called the latus-rectum of the parabola.

(b) $y^2 = -4ax$

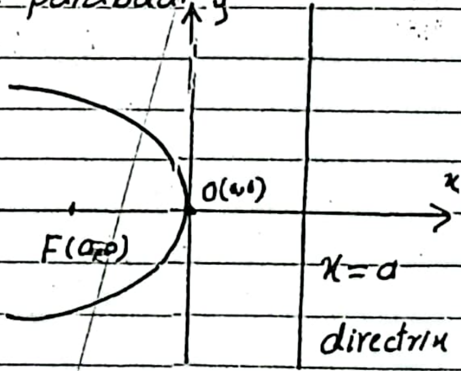
(1) Focus is at $F(-a,0)$

(2) Vertex is at $O(0,0)$

(3) x-axis is the axis of parabola

(4) Length of Latus-rectum $= 4a$

(5) Equation of the directrix is $x = a$

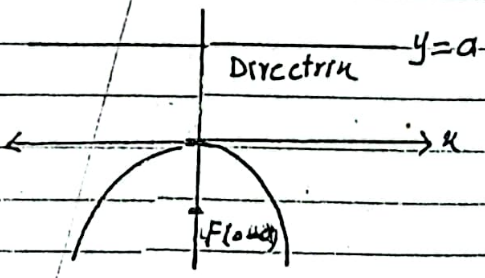


(C)# $x^2 = -4ay$

(1) Focus is at $F(0,-a)$

(2) Vertex is at $O(0,0)$

(3) y-axis is the axis



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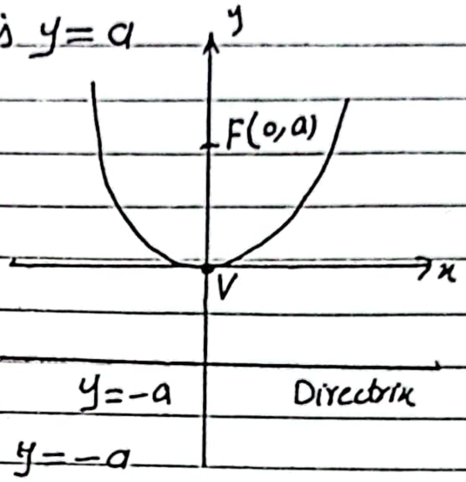
and its equation is $x=0$

(4) Length of the Latus Rectum $= 4a$

(5) Equation of the directrix is $y=a$

$x^2 = 4ay$



(d) # (1) Focus is at $F(0,a)$

(2) Vertex is at $O(0,0)$

(3) y-axis is the axis of parabola

(4) Length of Latus-Rectum $= 4a$

(5) Equation of directrix is $y=-a$

Note # Distance of focus from vertex = Distance of directrix from vertex $= a = \frac{1}{4}$(Length of L.R)

General Equations of Parabola

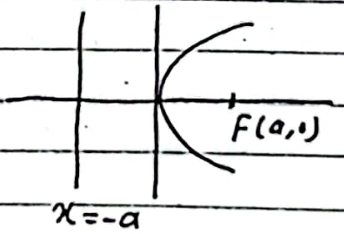
(a) Let the vertex of parabola be at (h,k), then

$$(y-k)^2 = 4a(x-h)$$

$$Y^2 = 4aX \rightarrow ① \text{ where } Y = y-k,\ X = x-h$$

Comparing it with standard parabola

$$y^2 = 4ax \rightarrow ②$$



Vertex

Vertex is at

$X=0$ | $Y=0$

$\Rightarrow x-h=0$ | $y-k=0$

$\Rightarrow x=h$ | $y=k$

Vertex $=(h,k)$

Directrix #

Equation of directrix is

$X=-a \Rightarrow x-h=-a \Rightarrow x=h-a$

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Axis of Parabola

Equation of the axis of parabola is

$Y = 0$

$\Rightarrow y - k = 0 \Rightarrow y = k$ which is a line parallel to x-axis

Focus

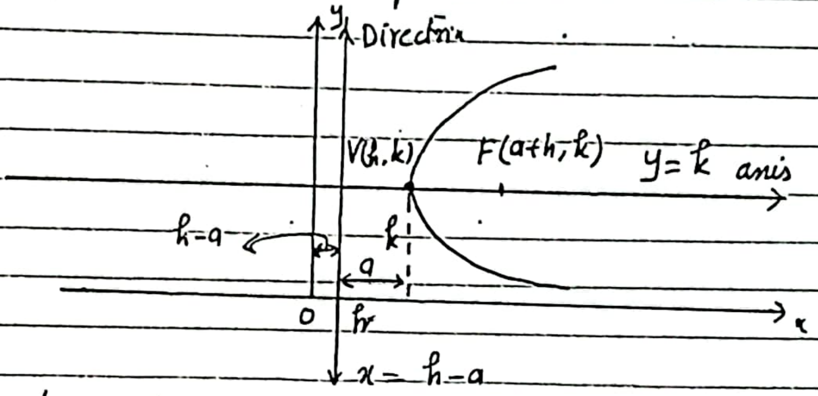
Co-ordinates of focus are

$X = a$ $\quad$ $Y = 0$

$x - h = a$ $\quad$ $y - k = 0$

$x = h + a$ $\quad$ $y = k$

Hence parabola has shape as under.



Length of Latus-rectum = 4a.

Similarly we can trace out the following general parabolas by comparing these with the corresponding standard parabolas

(1) $(y-k)^2 = -4a(x-h)$ $\quad$ (2) $(x-h)^2 = 4a(y-k)$

(3) $(x-h)^2 = -4a(y-k)$

More general equation of Parabola

Any 2nd degree equation of the form

$ax^2 + by^2 + 2gx + 2fy + c = 0$ represents a parabola if

either $a = 0$, $b \neq 0$ or $a \neq 0$, $b = 0$

e.g. $x^2 + 4x + 2y - 7 = 0$ is an equation of parabola

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Problem # Find the vertex, focus, directrix, axis and latus-rectum of the parabola

$$x^2+4x+2y-7=0$$

Sol #

$$x^2+4x+2y-7=0$$

$$\Rightarrow x^2+4x = -2y+7$$

adding 4 to both sides

$$x^2+4x+4 = -2y+11$$

$$(x+2)^2 = -2\left(y-\frac{11}{2}\right)$$

$$X^2 = -2Y \rightarrow (1)$$

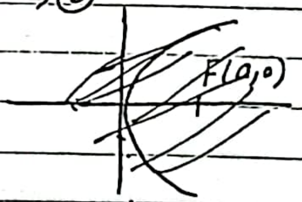
where $X = x+2$ $\quad Y = y-\frac{11}{2}$

Comparing (1) with

$$X^2 = -4aY \rightarrow (2)$$

$$-4a = -2$$

$$a = +\frac{1}{2}$$



Vertex

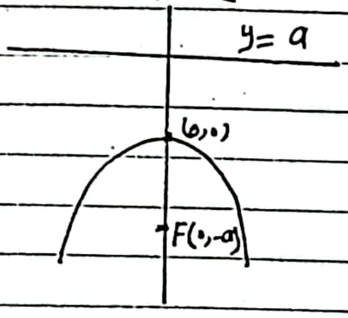
Vertex is given by

$X=0$ $\qquad Y=0$

$x+2=0$ $\qquad y-\frac{11}{2}=0$

$x=-2$ $\qquad y=\frac{11}{2}$

Hence Vertex $=\left(-2,\frac{11}{2}\right)$



Focus

focus is given by

$X=0$ $\qquad Y=-a$

$x+2=0$ $\qquad y-\frac{11}{2}=-\frac{1}{2}$

$x=-2$ $\qquad y=5$

$\therefore$ The focus $=(-2,5)$

Equation of axis #

Equation of axis is given by

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$X=0 \Rightarrow x+2=0 \Rightarrow x=-2.$

The length of the latus rectum $= 2$

Remarks # (1) When the axis of parabola is parallel to one of the Co-ordinate axes, the equation of the parabola does not contain any product term of the form xy and it can be transferred to one of the forms $y^2=4ax$ or $x^2=4ay$ by suitably shifting the origin.

(2) When the axis of the parabola is not parallel to any of the Co-ordinate axes, the equation of parabola contains the product term xy and it cannot be tranformed to one of the standard forms by merely shifting the origin; it is also necessary to turn the axis suitably.

Projectile

A particle which is projected freely with an initial velocity in a direction oblique to the direction of gravity is called a projectile.

OR

A particle projected in such a way that it covers horizontal and vertical distances, (is called a) under the action of its own weight is called a projectile.

OR

A particle projected with an initial velocity making some oblique angle with the vertical and moves freely under the action of its own weight is called a projectile.

Remarks # If the initial velocity of a projectile is vertical it moves in a straight line.

If the initial velocity is not verticle the particle moves in a curve and its flight can be analyzed by considering the vertical and horizontal components of its acc, velocity and displacement.

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examples of projectile are bullet fired from gun, bomb dropped from a plane, a ball thrown etc.

Trajectory of Projectile

The path followed by the projectile is called the trajectory of projectile.

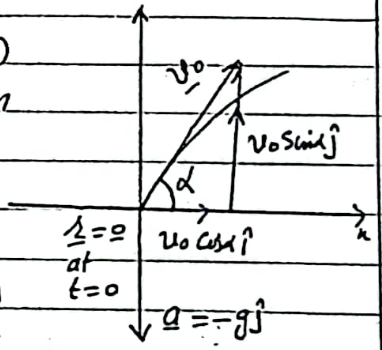
Lanch angle or firing angle or angle of elevation

It is the angle which the initial velocity of projectile makes with the horizontal

Equation of motion of an ideal Projectile

Suppose a projectile is launched in a vertical plane from a point O with velocity v_0. We take the point of projection O as the origin and let the horizontal line ox and the vertically upwards line oy be taken as axes. Suppose the projectile is launched in the 1st quadrant and its initial velocity v_0 makes angle α with horizontal. Then

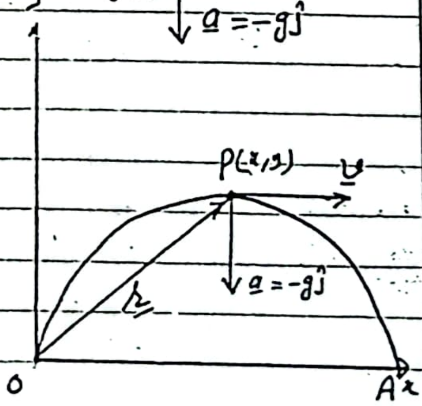
$$\vec{v}_0 = v_0 \cos\alpha\, \hat{i} + v_0 \sin\alpha\, \hat{j} \longrightarrow (1)$$



Let at any time t after the projection P(x,y) be the position of the projectile such that

$$\vec{OP} = \vec{r} = x\hat{i} + y\hat{j} \longrightarrow (2)$$

If air resistance is neglected and m be the mass of projectile, then the only force acting on the projectile is its weight mg acting vertically downward. Therefore equation of motion is



$$\vec{F} = m\vec{a} = -mg\hat{j}$$

$$\vec{a} = -g\hat{j}$$

$$\Rightarrow \frac{d^2\vec{r}}{dt^2} = -g\hat{j}$$

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Integrating

$$\int \frac{d^2 \vec{r}}{dt^2} dt = -\int g \hat{j}\, dt$$

$$\vec{v} = \frac{d\vec{r}}{dt} = -gt\hat{j} + \vec{A} \longrightarrow (3)$$

where $\vec{A}$ is constant of integration

Initially at $t=0$ $\vec{v} = \vec{v}_0$

$$\Rightarrow \vec{v}_0 = 0 + \vec{A}$$

using in (3)

$$\vec{v} = -gt\hat{j} + \vec{v}_0 \longrightarrow (4)$$

$$= -gt\hat{j} + v_0 \cos\alpha\, \hat{i} + v_0 \sin\alpha\, \hat{j}$$

$$v_x \hat{i} + v_y \hat{j} = v_0 \cos\alpha\, \hat{i} + (v_0 \sin\alpha - gt)\hat{j}$$

Comparing co-efficients of $\hat{i}, \hat{j}$ on both sides

$$\left.\begin{aligned} v_x &= v_0 \cos\alpha \\ v_y &= v_0 \sin\alpha - gt \end{aligned}\right\} \longrightarrow (5)$$

These equations give the components of vel in the horizontal and vertical directions. We note that for a given initial velocity and angle of projection horizontal velocity does not depend upon time and remain constant throughout the flight but vertical component of velocity depends upon time t and decreases by factor gt with time. At the peack point of the path of motion whole of the vel is horizontal because vertical component of vel becomes zero

Horizontal & Vertical distances at any Time t

from Equation (4)

$$\vec{v} = -gt\hat{j} + \vec{v}_0$$

$$\Rightarrow \frac{d\vec{r}}{dt} = -gt\hat{j} + \vec{v}_0$$

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$$\int d\underline{r} = \int(-gt\hat{j} + \underline{v_0})\, dt$$

$$\underline{r} = \underline{v_0}t - \frac{1}{2}gt^2\hat{j} + \underline{B} \rightarrow (6)$$

at $t=0$ $\quad \underline{r} = \underline{0}$

$\Rightarrow \underline{0} = \underline{0} + \underline{0} + \underline{B}$

$\Rightarrow \underline{B} = \underline{0}$

using in (6)

$$\underline{r} = \underline{v_0}t - \frac{1}{2}gt^2\hat{j}$$

But $\underline{r} = x\hat{i} + y\hat{j}$

$$\Rightarrow x\hat{i} + y\hat{j} = (v_0\cos\alpha\,\hat{i} + \hat{j}\,V_0\sin\alpha)t - \frac{1}{2}gt^2\hat{j}$$

$$= V_0\cos\alpha\, t\,\hat{i} + (V_0\sin\alpha\, t - \frac{1}{2}gt^2)\hat{j}$$

Comparing Co-efficients of $\hat{i}, \hat{j}$

$$x = V_0\cos\alpha\, t \rightarrow (7)$$

$$y = t\,V_0\sin\alpha - \frac{1}{2}gt^2 \rightarrow (8)$$

Equations (7) & (8) give horizontal and vertical distances covered in time t. These equations are also called parametric equations of the curve (path of projectile) and time t is parameter.

Equation of the path of projectile in cartesian form can be obtained by eliminating t from (7) & (8)

from (7)

$$t = \frac{x}{v_0\cos\alpha}$$

using this value in (8) equation of the path is

$$y = V_0\sin\alpha\left(\frac{x}{v_0\cos\alpha}\right) - \frac{1}{2}g\left(\frac{x}{v_0\cos\alpha}\right)^2$$

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$$y = x\tan\alpha - \frac{1}{2}\frac{gx^2}{v_0^2\cos^2\alpha} \longrightarrow (9)$$

Nature of The Trajectory

We prove that trajectory of projectile is a parabola. Equation of motion of a projectile projected in a vertical plane through the point of projection with initial vel v_0 making angle α with the horizontal level in 1st quadrant is given by

$$y = x\tan\alpha - \frac{gx^2}{2v_0^2\cos^2\alpha}$$

$$\Rightarrow 2v_0^2\cos^2\alpha\, y = x\tan\alpha \cdot 2v_0^2\cos^2\alpha - gx^2$$

$$x^2 - x\frac{2v_0^2}{g}\sin\alpha\cos\alpha = -\frac{2v_0^2\cos^2\alpha}{g}y$$

Adding $\frac{v_0^4\sin^2\alpha\cos^2\alpha}{g^2}$ to both side to

Complete square on L.H.S

$$x^2 - \frac{2v_0^2\sin\alpha\cos\alpha}{g} + \frac{v_0^4\sin^2\alpha\cos^2\alpha}{g^2} = -\frac{2v_0^2\cos^2\alpha}{g}y + \frac{v_0^4\sin^2\alpha\cos^2\alpha}{g^2}$$

$$\Rightarrow \left(x - \frac{v_0^2\sin\alpha\cos\alpha}{g}\right)^2 = -\frac{2v_0^2\cos^2\alpha}{g}\left(y - \frac{v_0^2\sin^2\alpha}{2g}\right) \longrightarrow (10)$$

which is an equation of parabola with vertex

$$V\left(\frac{v_0^2\sin\alpha\cos\alpha}{g}, \frac{v_0^2\sin^2\alpha}{2g}\right)$$

and latus-rectum $= \frac{2v_0^2\cos^2\alpha}{g}$

[This can remembered as the reciprocal of the co-efficients of x^2 in equation(9)]

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Writing ⑩ in the form.

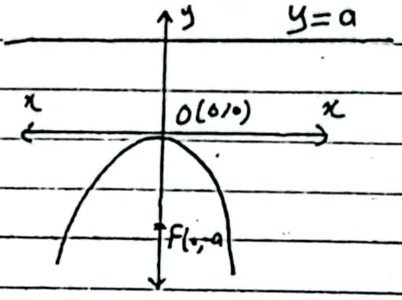
$$X^2 = -\frac{2v_0^2\cos^2\alpha}{g}Y \longrightarrow (11)$$

where $X = x - \frac{v_0^2\sin\alpha\cos\alpha}{g}$

$$Y = y - \frac{v_0^2\sin^2\alpha}{2g}$$

Comparing ⑪ with standard Equation of Parabola

$$x^2 = -4ay \longrightarrow (12)$$



Equation of axis #

$$X = 0$$

$$\Rightarrow x - \frac{v_0^2\sin\alpha\cos\alpha}{g} = 0$$

$$x = \frac{v_0^2\sin\alpha\cos\alpha}{g}$$

Hence equation of the axis of parabola is

$$x = \frac{v_0^2\sin\alpha\cos\alpha}{g}$$ which is parallel to y-axis

Also

$$-\frac{2v_0^2\cos^2\alpha}{g} = -4a$$

$$\Rightarrow a = \frac{v_0^2\cos^2\alpha}{2g} \longrightarrow (13)$$

Focus #

Co-ordinates of the focus are given by

$X = 0$ & $Y = -a$

$\Rightarrow x - \frac{v_0^2\sin\alpha\cos\alpha}{g} = 0$; $y - \frac{v_0^2\sin^2\alpha}{2g} = -\frac{v_0^2\cos^2\alpha}{2g}$

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$$x = \frac{v_0^2 \sin\alpha \cos\alpha}{g} \qquad y = \frac{v_0^2 \sin^2\alpha}{2g} - \frac{v_0^2 \cos^2\alpha}{2g}$$

$$= \frac{2v_0^2 \sin\alpha \cos\alpha}{2g} \qquad y = -\frac{v_0^2}{2g}\left(\cos^2\alpha - \sin^2\alpha\right)$$

$$= \frac{v_0^2 \sin 2\alpha}{2g} \qquad y = -\frac{v_0^2 \cos 2\alpha}{2g}$$

Thus focus $= F\left(\frac{v_0^2 \sin 2\alpha}{2g}, -\frac{v_0^2 \cos 2\alpha}{2g}\right)$.

When $\alpha = \pi/4$ focus is

$F\left(\frac{v_0^2}{2g}, 0\right) \Rightarrow$ focus lies on the horizontal line. Thus focus lies above, on or below the horizontal line Ox according as angle of projection α is greater than, equal to or less than $\pi/4$. In case of maximum range, the focus lies in the range.

Height of Directrix OR Equation of directrix

Equation of directrix is given by

$$Y = a$$

$$y - \frac{v_0^2 \sin^2\alpha}{2g} = \frac{v_0^2 \cos^2\alpha}{2g}$$

$$\Rightarrow y = \frac{v_0^2 \cos^2\alpha}{2g} + \frac{v_0^2 \sin^2\alpha}{2g}$$

$$= \frac{v_0^2}{2g}\left(\sin^2\alpha + \cos^2\alpha\right)$$

$$y = \frac{v_0^2}{2g}$$ which is required equation of the directrix

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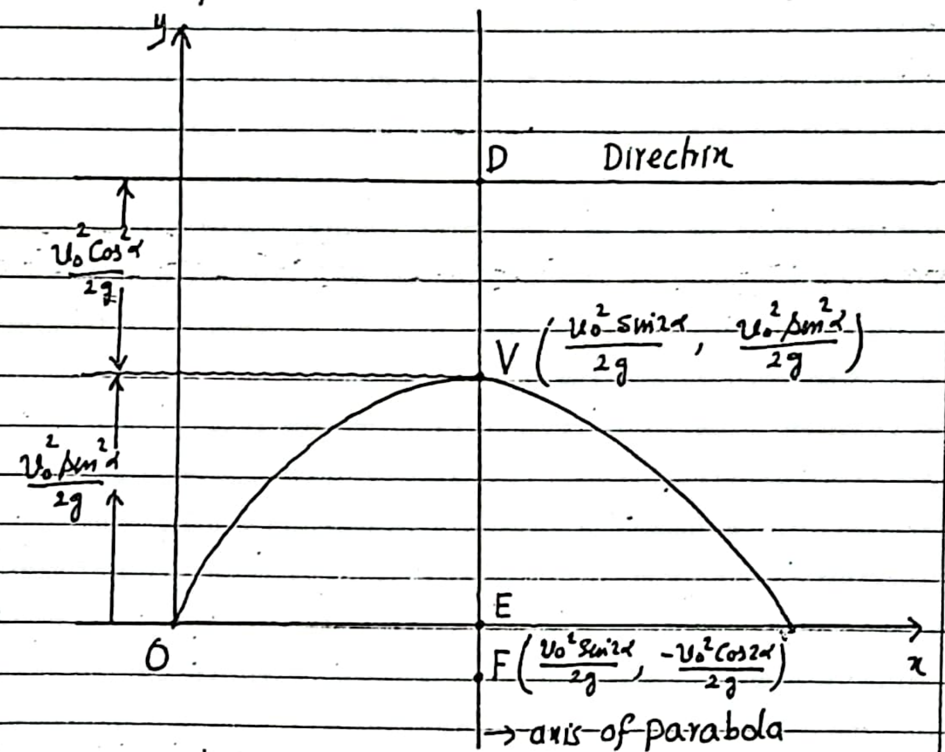
It also shows that height of the directrix above the level of projection is $\frac{v_0^2}{2g}$ and it depends upon the initial velocity v_0 of the projectile and is independent of the angle of projection.



The Greatest Height

As the vertex of the trajectory of the projectile is the highest point from the x-axis, therefore

H = Maximum height = ordinate of the vertex

$$\boxed{H = \frac{v_0^2 \sin^2\alpha}{2g}}$$

OR

The greatest height can also be calculated by considering vertical motion as

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Initial vertical velocity $= V_o \sin\alpha = v_i$

Final vertical velocity at max height $= 0 = v_f$

acc $= -g$

Distance covered = Max height = H

$$2ax = v_f^2 - v_i^2$$

$$2(-g)H = 0 - V_o^2 \sin^2\alpha$$

$$H = \frac{V_o^2 \sin^2\alpha}{2g}$$

Time to Reach Maximum Height

We know that at the maximum height vertical component of velocity becomes zero. Hence time to reach max. height is the time during which vertical velocity of the projectile becomes zero.

Now vertical velocity at any time is given by

$$v_y = V_o \sin\alpha\, t - \frac{1}{2} g t^2$$

at maximum height $v_y = 0$

$$0 = V_o \sin\alpha\, t - \frac{1}{2} g t^2$$

$$0 = 2V_o \sin\alpha\, t - g t^2$$

$$= t(2V_o \sin\alpha - gt)$$

$\Rightarrow t = 0$ or $2V_o \sin\alpha - gt = 0$

at $= 0$ $\Rightarrow t = \frac{V_o \sin\alpha}{g}$

Thus time to reach maximum height is

$$\boxed{t = \frac{V_o \sin\alpha}{g}}$$

Total Time of Flight

It is the time which the projectile takes from the point of projection to reach

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again the level of projection.

When the projectile strikes the level of projection, its vertical distance is zero.

Now vertical distance at any time t is given by

$$y = v_o \sin\alpha\, t - \frac{1}{2}gt^2$$

putting $y = 0$

$$0 = v_o \sin\alpha\, t - \frac{1}{2}gt^2$$

$$0 = 2v_o \sin\alpha\, t - gt^2$$

$$0 = (2v_o \sin\alpha - gt)t$$

$\Rightarrow t = 0$ (at O) or $2v_o \sin\alpha - gt = 0$

$$\Rightarrow t = \frac{2v_o \sin\alpha}{g}$$

Hence total time of flight is

$$T = \frac{2v_o \sin\alpha}{g}$$

Horizontal Range

The horizontal distance described by the projectile during total time of flight is called the horizontal range.

OR

It is the distance between the point of projection and the final position on the horizontal plane through the point of projection.

Horizontal range = Horizontal distance in time $T = \frac{2v_o \sin\alpha}{g}$

= (Horizontal vel) (total time of flight)

$$= v_o \cos\alpha \cdot \frac{2v_o \sin\alpha}{g}$$

$$= \frac{2v_o^2 \sin\alpha \cos\alpha}{g}$$

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= twice the abscissa of the vertex

Thus Horizontal range is

$$R = \frac{2v_0^2 \sin\alpha \cos\alpha}{g}$$

$$\boxed{R = \frac{v_0^2 \sin 2\alpha}{g}}$$

OR

Range of the projectile can found by putting $y = 0$ and calculating horizontal distance by the equation of motion of projectile as.

Equation of motion of projectile is

$$y = x\tan\alpha - \frac{gx^2}{2V_0^2\cos^2\alpha}$$

putting $y = 0$

$$\Rightarrow 0 = x\tan\alpha - \frac{gx^2}{2V_0^2\cos^2\alpha}$$

$$\Rightarrow 0 = x\left(\tan\alpha - \frac{gx}{2V_0^2\cos^2\alpha}\right)$$

$\Rightarrow x = 0$ or $\tan\alpha - \frac{gx}{2v_0^2\cos^2\alpha} = 0$

Corresponds to initial pt.

$$x = \frac{2v_0^2\cos^2\alpha\tan\alpha}{g}$$

$$x = \frac{2v_0^2 \sin\alpha\cos\alpha}{g}$$

Horizontal range $= \frac{2v_0^2\sin\alpha\cos\alpha}{g} = \frac{v_0^2 \sin 2\alpha}{g}$

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Maximum Horizontal Range

The horizontal range R is given by

$$R = \frac{v_0^2 \sin 2\alpha}{g}$$

For a given value of v_0, the horizontal range depends upon $\sin 2\alpha$ and is max. when $\sin 2\alpha$ is maximum i.e when

$$\sin 2\alpha = 1$$

$$\Rightarrow 2\alpha = \frac{\pi}{2}$$

$$\alpha = \frac{\pi}{4}$$

Thus the maximum horizontal range with a given velocity v_0 is

$$R_{max} = \frac{v_0^2}{g}$$

$$\boxed{R_{max} = \frac{v_0^2}{g}}$$

Directions of Projection for a given velocity and for a given Horizontal Range

Since $R = \frac{v_0^2}{g} \sin 2\alpha = \frac{v_0^2}{g} \sin(\pi - 2\alpha)$

$$= \frac{v_0^2}{g} \sin 2\left(\frac{\pi}{2} - \alpha\right)$$

Therefor for a given horizontal range and with the same given velocity v_0, there are two directions of projection i.e α and $\frac{\pi}{2} - \alpha$. These two directions are such that the inclination of one with the horizontal is equal to the inclination of the other with vertical. We show that these two directions of projections for the same range are equally inclined with the direction of maxamum

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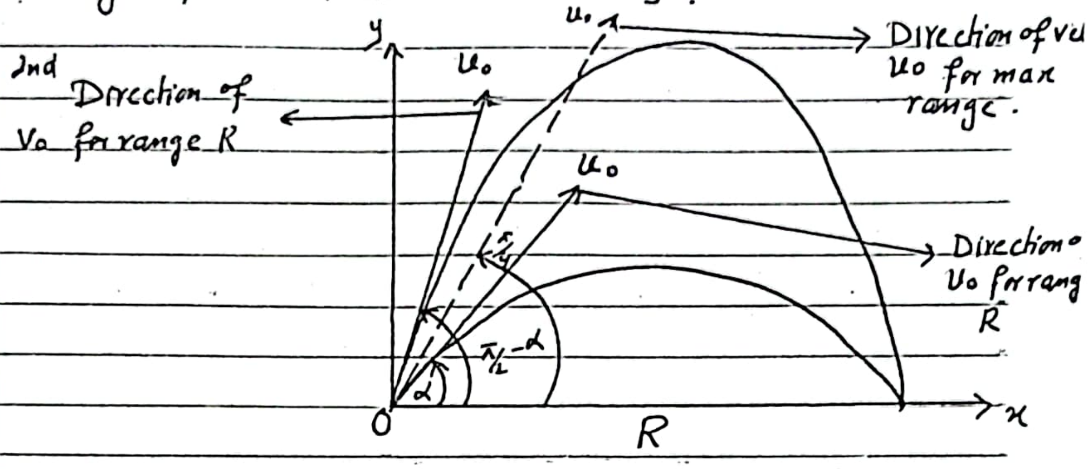
range for the same initial velocity v_0



Angle between the direction of maximum range and 1st direction (i.e direction α) for range $R = \frac{\pi}{4} - \alpha$

Aangle between the direction of maximum rangle and 2nd direction of projection for range $R = \frac{\pi}{2} - \alpha - \pi/4$

$= \pi/4 - \alpha$.

Thus we note that the two directions of projection for the same range R with same velocity v_0 are equally inclined with the direction of v_0 for maximum range or direction of maximum range bisects the angle between two directions for the same horizontal range and with same velocity of projection.

Also if range is max with velocity v_0, then $\alpha = \pi/4$ and 2nd direction of projection $= \pi/2 - \alpha = \pi/2 - \pi/4 = \pi/4$

Thus for maximum range with a given velocity of projection v_0 there is only one direction of projection.

Direction of Motion of Projecte at any instant & at any Point

Direction of motion of a projectile can be found in two ways.

(1) Direction of motion at any instant (time) is given by

$$\frac{dy/dt}{dx/dt} = \frac{v_y}{v_x} = \frac{V_0 \sin\alpha - gt}{V_0 \cos\alpha}$$

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(2) Direction of motion at any point is given by $\frac{dy}{dx}$

Magnitude of the Velocity at any height from the Projection Leval

Horizontal and vertical velocities of the projectile projected with initial vel v_0 at an angle α are given by

$$v_x = v_0 \cos\alpha.$$
$$v_y = v_0 \sin\alpha - gt$$

We note that vertical component decreases with time while horizontal component remain same. Therefore magnitude of velocity changes with change in vertical velocity. Hence we find the magnitude of the velocity of projectile at any height above the level of projection. Let v be vel at point $P(x,y)$.

Now

$$|\underline{V}|^2 = v_x^2 + v_y^2$$
$$= (v_0\cos\alpha)^2 + (V_0\sin\alpha - gt)^2$$
$$= V_0^2\cos^2\alpha + V_0^2\sin^2\alpha + g^2t^2 - 2V_0 g t . \sin\alpha$$
$$= V_0^2 - 2g(v_0 \sin\alpha t - \tfrac{1}{2}gt^2)$$

But $y = v_0 \sin\alpha t - \frac{1}{2}gt^2$ is height at time t when net velocity is v. Thus

$$\boxed{V^2 = V_0^2 - 2gy} \longrightarrow (1)$$

which is required velocity at any height y from the projection level.

Deductions from (1)

(1) when $y=0$ i.e when

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Projectile is at level of projection initially or finally magnitude of its velocity is

$$V^2 = V_0^2$$

$$V = V_0$$

It means that projectile 1st losses velocity upto maximum height and then its velocity increases by the same amount which was lost when it strikes the projection level

(2) At maximum height when $y = \frac{V_0^2 \sin^2\alpha}{2g}$

, the magnitude of velocity is

$$V^2 = V_0^2 - 2g \cdot \frac{V_0^2 \sin^2\alpha}{2g}$$

$$= V_0^2 - V_0^2 \sin^2\alpha$$

$$= V_0^2 (1 - \sin^2\alpha)$$

$$= V_0^2 \cos^2\alpha$$

$$\Rightarrow V = V_0 \cos\alpha = \text{horizontal velocity}$$

~~It is logical because vertical velocity~~ at max. height is zero and whole of the velocity is horizontal velocity.

(3) When $y = \frac{1}{2}(\text{Maximum height})$

$$= \frac{1}{2} \cdot \frac{V_0^2 \sin^2\alpha}{2g}$$

velocity is given by

$$V^2 = V_0^2 - 2g \cdot \frac{V_0^2 \sin^2\alpha}{4g}$$

$$= V_0^2 - \frac{V_0^2 \sin^2\alpha}{2}$$

$$= V_0^2 \left[1 - \frac{\sin^2\alpha}{2}\right] = V_0^2 \left[\frac{2 - \sin^2\alpha}{2}\right]$$

$$= \frac{V_0^2}{2}\left[1 + 1 - \sin^2\alpha\right] = \frac{V_0^2}{2}\left[1 + \cos^2\alpha\right]$$

$$\boxed{V = \frac{V_0}{\sqrt{2}}\sqrt{1 + \cos^2\alpha}}$$

Which is the velocity of projectile when it has covered one half of its maximum Height.

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from ① we have

$$V^2 = 2g\left(\frac{v_0^2}{2g} - y\right) = 2g(\text{depth of pt. below The directrix})$$

$$= 2g(\text{Height of the directrix} - \text{ordinate of } P)$$

$$= 2g(\text{Height of the directrix above } P)$$

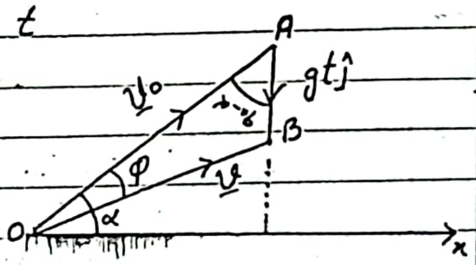
$$V = \sqrt{2g(\text{height of the drirectrix above } P)}$$

It follows that the magnitude of the velocity at any point of the path is the same as the magnitude of velocity acquired in falling freely from rest a vertical distance from the directrix to that point.

Geometrical Method of finding Velocity

The velocity of particle after t seconds of projection from O with velocity v_0 at an angle α above the horizontal, can also be obtained geometrically as follows.

Since velocity at any time t is given by



$$\underline{v} = \underline{v_0} - gt\hat{j}$$

$$= \underline{v_0} + gt(-\hat{j})$$

$\Rightarrow \underline{v}$ is the resultant of $\underline{v_0}$ and $gt(-\hat{j})$

Let $\overrightarrow{OA}$ respresents v_0 and $\overrightarrow{AB}$ vertically downwards represents $gt(-\hat{j})$. Then $\overrightarrow{OB}$ represents velocity v.

$$\angle XOA = \alpha$$

$$\angle OAB = 90^\circ - \alpha$$

Applying Law of Cosines to $\triangle OAB$

$$OB^2 = OA^2 + AB^2 - 2OA \cdot AB \cos\angle OAB$$

$$V^2 = v_0^2 + g^2t^2 - 2v_0gt\cos(90^\circ - \alpha)$$

$$= v_0^2 + g^2t^2 - 2v_0gt\sin\alpha$$

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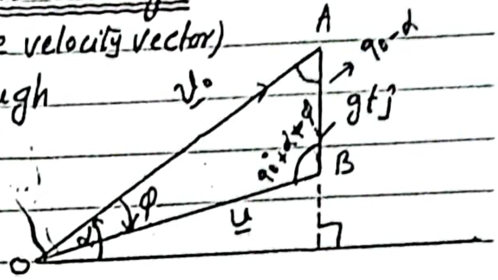
Change in the direction of motion and time Required for that Change

If the direction of motion (i.e velocity vector) of the particle has turned through an angle ϕ from the original direction (direction of initial velocity v_0)



$\angle AOB = \phi$

$\because \angle OAB = 90^\circ - \alpha$

$\therefore \angle OBA = 180^\circ - (90^\circ - \alpha + \phi)$

$= 90^\circ - \phi + \alpha$

$= 90^\circ + \alpha - \phi$

By applying law of sines to $\triangle OAB$

$$\frac{AB}{\sin\phi} = \frac{OA}{\sin[90^\circ + \alpha - \phi]}$$

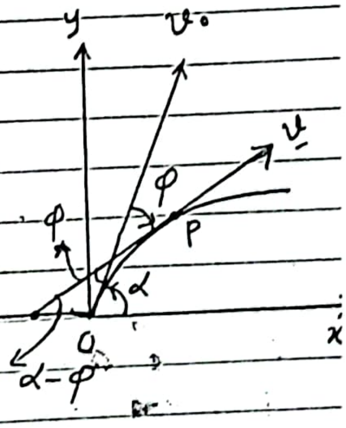
$$\Rightarrow \frac{gt}{\sin\phi} = \frac{v_0}{\cos(\phi - \alpha)}$$

so that the time required for change ϕ from the original direction is given by

$$\boxed{t = \frac{v_0 \sin\phi}{g\cos(\phi - \alpha)}}$$

OR

Suppose velocity $\underline{v}$ at point P has turned through an angle ϕ from the initial direction of initial velocity v_0 in time t. Then angle of v with x-axis is $\alpha - \phi$ and



$$\tan(\alpha - \phi) = \frac{v_y}{v_x}$$

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$$\tan(\alpha-\phi) = \frac{v_0 \sin\alpha - gt}{v_0 \cos\alpha}$$

$$= \frac{v_0 \sin\alpha}{v_0 \cos\alpha} - \frac{gt}{v_0 \cos\alpha}$$

$$= \tan\alpha - \frac{gt}{v_0 \cos\alpha}$$

$$\Rightarrow \frac{gt}{v_0 \cos\alpha} = \tan\alpha - \tan(\alpha-\varphi)$$

$$= \frac{\sin\alpha}{\cos\alpha} - \frac{\sin(\alpha-\varphi)}{\cos(\alpha-\varphi)}$$

$$= \frac{\sin\alpha \cos(\alpha-\varphi) - \cos\alpha \sin(\alpha-\varphi)}{\cos\alpha \cos(\alpha-\varphi)}$$

$$= \frac{\sin[\alpha-(\alpha-\varphi)]}{\cos\alpha \cos(\alpha-\varphi)}$$

$$= \frac{\sin\varphi}{\cos\alpha \cos(\alpha-\varphi)}$$

$$\Rightarrow t = \frac{v_0 \sin\varphi}{g \cos(\alpha-\varphi)}$$

In particular if the velocity is perpendicular to the original direction, then $\varphi = 90^\circ$ and we get

$$t = \frac{v_0 \sin 90^\circ}{g \cos(\alpha - 90^\circ)}$$

$$t = \frac{v_0}{g \sin\alpha}$$

Problem# A projectile is projected with a velocity v_0 at an elevation α. In what time direction of motion be $\alpha/2$

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Sol # Here we have to find the time during which direction has turned through an angle $\varphi = \alpha/2$ with original direction of motion. Let t be the required time, then

$$t = \frac{v_0 \sin\varphi}{g\cos(\alpha-\varphi)}$$

putting $\varphi = \alpha/2$

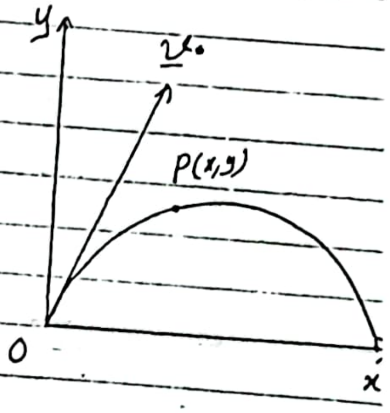
$$t = \frac{v_0 \sin\alpha/2}{g\cos(\alpha-\alpha/2)}$$

$$\boxed{t = \frac{v_0 \sin\alpha/2}{g\cos\alpha/2} = \frac{v_0}{g}\tan\alpha/2}$$

Velocity of Projectile in terms of Height

We have already found velocity in terms of height. Here we find it in a different method.

Let $P(x,y)$ be the position of projectile at any time t and $\underline{v}$ be its velocity at that point.



Now equation of motion of the projectile is

$$\frac{d\underline{v}}{dt} = -g\hat{j} \longrightarrow ①$$

Multiplying ① scalarly with $2\underline{v}$

$$2\underline{v}\cdot\frac{d\underline{v}}{dt} = -2g\,\underline{v}\cdot\hat{j}$$

$$\Rightarrow \frac{d}{dt}(\underline{v}\cdot\underline{v}) = -2g\left(\frac{dx}{dt}\hat{i}+\frac{dy}{dt}\hat{j}\right)\cdot\hat{j}$$

$$\Rightarrow \frac{d}{dt}v^2 = -2g\frac{dy}{dt}$$

$$\Rightarrow d(v^2) = -2g\,dy$$

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Integrating we have

$$v^2 = -2gy + c \longrightarrow (2)$$

where c is constant of integration

Initially at $t=0$, we have

$$v = v_0 \quad \& \quad y = 0$$

$$\Rightarrow \quad v_0^2 = c$$

putting this value of c in (2)

$$v^2 = v_0^2 - 2gy$$

$$\boxed{v = \sqrt{v_0^2 - 2gy}}$$

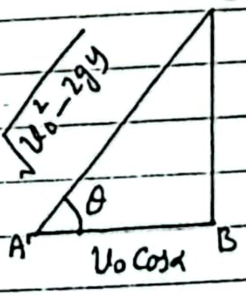
which is the required velocity of projectile at height y from projection level.

Further, if v makes angle θ with the horizontal, its horizontal component is given by

$$v\cos\theta = v_0\cos\alpha \quad (\because \text{horizontal vel remains constant})$$

$$\Rightarrow \cos\theta = \frac{v_0\cos\alpha}{v}$$

$$= \frac{v_0\cos\alpha}{\sqrt{v_0^2 - 2gy}}$$



$$\Rightarrow BC^2 = AC^2 - AB^2$$

$$= v_0^2 - 2gy - v_0^2\cos^2\alpha$$

$$= v_0^2(1 - \cos^2\alpha) - 2gy$$

$$= v_0^2\sin^2\alpha - 2gy$$

$$BC = \sqrt{v_0^2\sin^2\alpha - 2gy}$$

$$\Rightarrow \boxed{\tan\theta = \frac{\sqrt{v_0^2\sin^2\alpha - 2gy}}{v_0\cos\alpha}}$$

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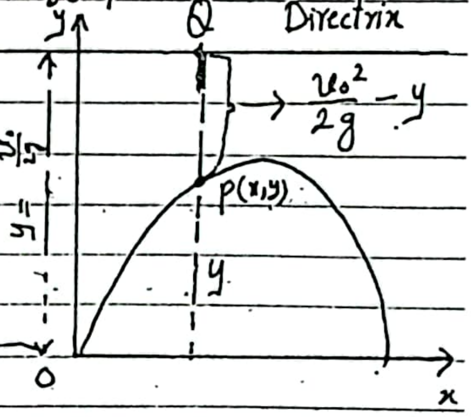
K.E of Projectile & Velocity of Projectile

Show that at any point P on the of a projectile the velocity (& kinetic energy) is the same as acquired by it in falling freely under gravity to P from that point on the directrix $y = \frac{v_0^2}{2g}$ which is vertically above P

OR

Show that speed at any point P of parabolic path is the same as obtained by the particle in falling freely from rest from a point on the directrix vertically above P to P.



Let v_0 be the initial velocity of projection and v be the velocity of projectile at point $P(x, y)$.

According to Law of of Conservation of energy

Total initial energy = Total final energy

$$\frac{1}{2}mv_0^2 + 0 = \frac{1}{2}mv^2 + mgy$$

$$\Rightarrow \frac{1}{2}mv^2 = \frac{1}{2}mv_0^2 - mgy \longrightarrow ①$$

$$\text{K.E at point P} = \frac{1}{2}mv_0^2 - mgy \longrightarrow ②$$

Now let the projectile falls freely from rest to point P from a point Q on the directrix vertically above P. Let v' be the velocity attained at point P. Then again by Law of Conservation

Total initial energy = Total final energy.

$$mg \cdot \frac{v_0^2}{2g} + 0 = \frac{1}{2} m v'^2 + mgy$$

$$\frac{1}{2} m v_0^2 = \frac{1}{2} m v'^2 + mgy$$

$$\Rightarrow \frac{1}{2} m v'^2 = \frac{1}{2} m v_0^2 - mgy \longrightarrow (3)$$

K.E attained at P in falling freely from rest from Q to P

$$= \frac{1}{2} m v_0^2 - mgy \longrightarrow (4)$$

∴ From (2) & (4), we have

K.E of projectile at P = K.E at P of the projectile attained in falling freely from a point Q on directrix vertically above P

Also from (1)

$$\frac{1}{2} m v^2 = \frac{1}{2} m v_0^2 - mgy$$

$$\Rightarrow v^2 = v_0^2 - 2gy$$

$$\Rightarrow (\text{Velocity at } P)^2 = v^2 = v_0^2 - 2gy \longrightarrow (5)$$

from (3)

$$\frac{1}{2} m v'^2 = \frac{1}{2} m v_0^2 - mgy$$

$$\Rightarrow v'^2 = v_0^2 - 2gy \longrightarrow (6)$$

from (5) and (6)

$$v^2 = v'^2$$

$$\Rightarrow v = v'$$

⇒ vel of projectile at point P = vel of particle gained in falling freely from rest from Q to point P

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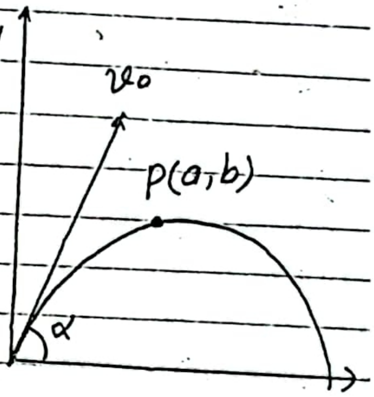
Directions and velocity of Projection For a Projectile to hit or to Pass a given Point

Suppose a projectile is launched from O with initial vel of Projection v_0 at an angle α with the horizontal. Then the equation of the trajectory is given by



$$y = x\tan\alpha - \frac{gx^2}{2v_0^2\cos^2\alpha} \rightarrow (1)$$

Let the projectile hit a given point (a,b), then point (a,b) will lie on (1) and we have

$$b = a\tan\alpha - \frac{ga^2}{2v_0^2\cos^2\alpha}$$

$$2bv_0^2 = 2a\tan\alpha . v_0^2 - ga^2\sec^2\alpha$$

$$2bv_0^2 = 2a\tan\alpha . v_0^2 - ga^2(1+\tan^2\alpha)$$

$$= 2a\tan\alpha . v_0^2 - ga^2 - ga^2\tan^2\alpha$$

$\Rightarrow$

$$ga^2\tan^2\alpha - 2av_0^2\tan\alpha + ga^2 + 2bv_0^2 = 0 \rightarrow (2)$$

This equation is quadratic in $\tan\alpha$ and has two roots in general i.e two directions of projections with a given velocity v_0 to hit a point (a,b). The roots this equation may be real and equal, real and distinct and imaginary depending on the initial velocity of projection

Case I # The roots of equation will be real (real directions) and distinct (two different directions) if

Disc: of ② is greater than zero i.e

$[-2av_0^2]^2 - 4ga^2(ga^2+2bv_0^2) > 0$

$\Rightarrow 4a^2v_0^4 - 4ga^2(ga^2+2bv_0^2) > 0$

$\Rightarrow v_0^4 - g(ga^2+2bv_0^2) > 0$

$\Rightarrow v_0^4 - g^2a^2 - 2bgv_0^2 > 0$

$\Rightarrow v_0^4 - 2bgv_0^2 > g^2a^2$

Adding b^2g^2

$v_0^4 - 2bgv_0^2 + b^2g^2 > g^2a^2 + g^2b^2$

$(v_0^2 - gb)^2 > g^2(a^2+b^2)$

$\Rightarrow v_0^2 - gb > g\sqrt{a^2+b^2}$

$\Rightarrow v_0^2 > gb + g\sqrt{a^2+b^2}$

$\Rightarrow v_0^2 > g(b+\sqrt{a^2+b^2})$

Thus when $v_0^2 > g(b+\sqrt{a^2+b^2})$, there will be two possible directions (angles) of projections for a projectile to pass through $P(a,b)$

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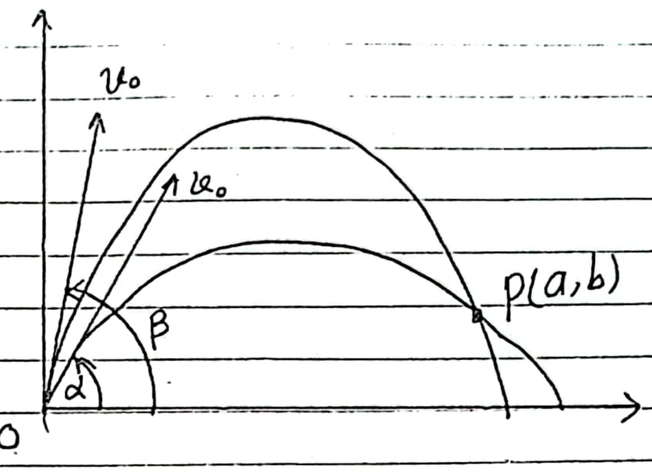

Case-II # The roots of equation ② will be real and equal (i.e there will be one one direction of projection to hit point (a,b)) if

Disc of ② $= 0$

if $4a^2v_0^4 - 4ga^2(ga^2+2bv_0^2) = 0$

$$\Rightarrow v_0^4 - g^2a^2 - 2bg\,v_0^2 = 0$$

$$\Rightarrow v_0^4 - 2bgv_0^2 = g^2a^2$$

$$\Rightarrow v_0^4 - 2bgv_0^2 + g^2b^2 = g^2a^2 + g^2b^2$$

$$\Rightarrow (v_0^2 + gb)^2 = g^2(a^2+b^2)$$

$$\Rightarrow v_0^2 - gb = g\sqrt{a^2+b^2}$$

$$\Rightarrow v_0^2 = g(b + \sqrt{a^2+b^2})$$

In this case value of v_0 is minimum to hit the point (a,b) and this min velocity is given by

$$\min v_0 = \boxed{\sqrt{g(b+\sqrt{a^2+b^2})}}$$

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Case-III # If disc of ② is $-ve$, there will be no real roots i.e no real direction to hit the point (a,b) and there will be no possible path to hit (a,b) with given initial vel v_0

Result # Thus if a particle is projected from a point Q to hit a point $P(a,b)$, then with a given velocity of projection there may be two possible paths, one possible path or no possible path.

Cor # If $OP = r$ and it makes angle α with the horizon, then

$$\sqrt{a^2+b^2} = r \quad \text{and} \quad b = r\sin\alpha$$

$$(\text{min Velocity})^2 = gr(1+\sin\alpha)$$

Times to hit a given Point

Suppose that the velocity of projection v_0 is such that the projectile pass through point (a,b) moving along two different paths. Then since (a,b) lies on the trajectory, therefor

$$a = v_0\cos\alpha\, t \longrightarrow ①$$

$$b = v_0\sin\alpha\, t - \frac{1}{2}gt^2 \longrightarrow ②$$

Eliminating α from ① & ②
from ②

$$b + \frac{1}{2}gt^2 = v_0\sin\alpha\, t \longrightarrow ③$$

squaring and adding ① & ③

$$\left(b+\frac{1}{2}gt^2\right)^2 + a^2 = v_0^2t^2 \longrightarrow (a)$$

$$b^2 + \frac{1}{4}g^2t^4 + bgt^2 + a^2 - v_0^2t^2 = 0$$

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$$\frac{1}{4}g^2t^4 + (bg - v_0^2)t^2 + a^2 + b^2 = 0$$

$$\Rightarrow t^4 + \frac{4}{g^2}(bg - v_0^2)t^2 + \frac{4(a^2+b^2)}{g^2} = 0$$

which is quadratic in t^2 and will have two roots. If t_1, t_2 are two roots, then

Product of roots $= t_1^2 t_2^2 = \dfrac{4(a^2+b^2)}{g^2}$

$$t_1^2 t_2^2 = \frac{4r^2}{g^2} \quad \text{where } r = OP = \sqrt{a^2+b^2}$$

$$\Rightarrow t_1 t_2 = \frac{2r}{g}$$

Hence the product of the two times of transits in two different paths is independent of the initial velocity

Also since $r = \frac{1}{2} g t_1 t_2 \longrightarrow (4)$

Now suppose a particle fall from rest vertically through a distance r. Then

Initial vel $= u = 0$

acc. $= g$

Let time to fall distance $r = t$

Equation of motion is

$$x = ut + \frac{1}{2}gt^2$$

$$r = 0t + \frac{1}{2}gt^2$$

$$r = \frac{1}{2}gt^2 \longrightarrow (5)$$

By comparing (4) & (5), it follows that the product of the two times of transits of projectile through a given point is equal to the square of the time taken by a particle in falling from rest vertically through a distance r (the distance of the given point from the point of projection)

If initial velocity of projection of projectile is such that it has only one direction (one path) to hit point (a,b),

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, then $t_1 = t_2 = t$ and from ④, we have

$$r = \frac{1}{2}gt^2$$

$$\Rightarrow t = \sqrt{\frac{2r}{g}}$$

This will be the time taken by projectile to hit point (a,b) with min initial velocity to hit the point (a,b).

Corollary # Replacing a by x and b by y in equation (a), we get

$$\left(y + \frac{1}{2}gt^2\right)^2 + x^2 = v_0^2 t^2$$

This equation is independent of directions of projection provided velocity of projection remains same.

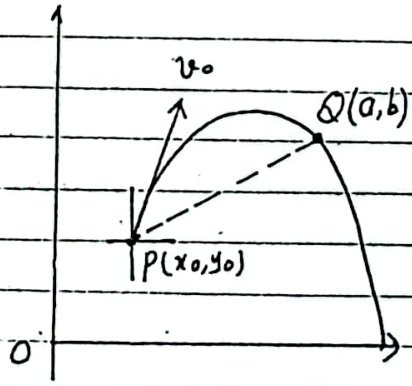
This shows that the particles projected simultaneously from the same point in different directions with the same initial velocity in a vertical plane will all lie on a circle after a time t.

Corollary # The product of the two times of flight from point P to Q with a given velocity of projection is $\frac{2PQ}{g}$.

Proof taking O at P & P at Q we have as above

$$t_1 t_2 = \frac{2OP}{g} = \frac{2PQ}{g}$$

OR



Suppose the projectile is launched from P with vel of projection v_0 making angle α with horizontal, then equations of motion are

$$x = x_0 + v_0 \cos\alpha\, t \rightarrow ①$$

$$y = y_0 + v_0 \sin\alpha\, t - \frac{1}{2}gt^2 \rightarrow ②$$

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$\because$ the projectile passes Through $Q(a,b)$, therefore

$$a = x_0 + V_0 \cos\alpha\, t \rightarrow (3)$$

$$b = y_0 + V_0 \sin\alpha\, t - \frac{1}{2} g t^2 \rightarrow (4)$$

$$\Rightarrow a - x_0 = V_0 \cos\alpha\, t \rightarrow (5)$$

and $(b-y_0) + \frac{1}{2} g t^2 = V_0 \sin\alpha\, t \rightarrow (6)$

squaring and adding (5) & (6)

$$(a-x_0)^2 + [(b-y_0) + \frac{1}{2} g t^2]^2 = v_0^2 t^2$$

$$(a-x_0)^2 + (b-y_0)^2 + \frac{1}{4} g^2 t^4 + (b-y_0) g t^2 - v_0^2 t^2 = 0$$

$$\Rightarrow \frac{1}{4} g^2 t^4 + \{(b-y_0) g - v_0^2\} t^2 + (a-x_0)^2 + (b-y_0)^2 = 0$$

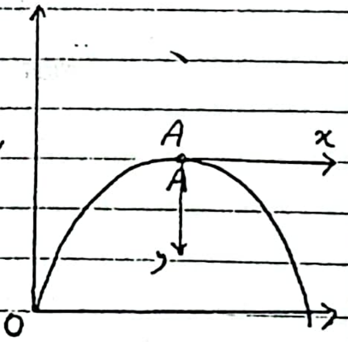
$$t^4 + \frac{4}{g^2}\{(b-y_0) g - v_0^2\} t^2 + \frac{4(PQ)^2}{g^2} = 0$$

If t_1, t_2 are its roots, then

$$t_1^2 t_2^2 = \frac{4(PQ)^2}{g^2}$$

$$t_1 t_2 = \frac{2PQ}{g} \quad \text{(proved)}$$

Highest Point as Origin



The solution to a problem is sometimes very much simplified by writing the equation of the trajectory w.reference to the highest point A as origin.

If v is the velocity of projectile at A, the horizontal line at A as x-axis and if the vertical line drawn downward be the y-axis, then Co-ordinates (x,y) of any point on the path of particle after time t are

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given by

$$x = vt \qquad y = \frac{1}{2}gt^2$$

and equation of path becomes

$$y = \frac{gx^2}{2v^2}$$

Projectile launched from a Point which is itself moving with uniform speed

Case I When a particle is projected from a platform or battleship or car etc moving forward to the target and initial velocity makes an angle α with line in which the plateform or ship or car etc is moving, then there is no change in the vertical component except the horizontal component of velocity relative to horizontal plane

Vertical component of vel $= V_0 \sin\alpha - gt \rightarrow (1)$

Horizontal component of vel $= V_0 \cos\alpha + V \rightarrow (2)$

(1) and (2) are parametric equations of trajectory.

time of flight $= T = \dfrac{2V_0 \sin\alpha}{g}$

Range $= R' =$ (Horizontal vel) (time of flight)

$$= (V_0\cos\alpha + V)\left(\frac{2V_0 \sin\alpha}{g}\right)$$

$$= \frac{V_0^2 \sin 2\alpha}{g} + \frac{2V_0 V \sin\alpha}{g}$$

If the particle is projected from a stationary point, then

Range $= R = \dfrac{V_0^2 \sin 2\alpha}{g}$

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Increase in range $= \frac{2 V_0 V \sin\alpha}{g}$

Case II # When particle is projected, making an angle α with horizontal line in which platform etc is moving backward

There will be change in the horizontal component of velocity of projectile relative to horizontal plane i.e

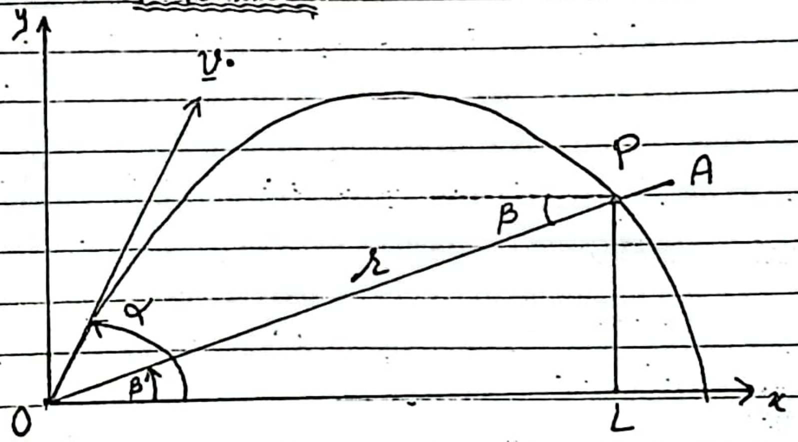
horizontal vel $= V_0 \cos\alpha - V$

$$\text{Range} = (V_0\cos\alpha - V)\frac{2V_0 \sin\alpha}{g}$$

$$= \frac{V_0^2 \sin 2\alpha}{g} - \frac{2 V V_0 \sin\alpha}{g}.$$

Decrease in range $= \frac{2 V V_0 \sin\alpha}{g}$

The Range of Projectile on an inclined Plane



Consider an inclined plane of inclination β to the horizontal. Let OA be the line of the greatest slope of the inclined plane.

Suppose a projectile is projected from point O in the vertical plane through OA (The vertical plane

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through O and inclined plane intersect in the line OA, with equation $y = x\tan\beta$). Let v_0 be the velocity of projection making an angle $\alpha > \beta$ to the horizontal.

If the projectile strikes the inclined plane at point P, then the distance $OP = r$ is called the range on the inclined plane.

The Co-ordinates of point P are.

$(r\cos\beta, r\sin\beta)$

The equation of the path of the projectile is.

$$y = x\tan\alpha - \frac{gx^2}{2v_0^2\cos^2\alpha} \longrightarrow (1)$$

Since the projectile passes through point P, therefore the Co-ordinates of P must satisfy (1) and we have

$$r\sin\beta = r\cos\beta\tan\alpha - \frac{gr^2\cos^2\beta}{2v_0^2\cos^2\alpha}$$

$$\Rightarrow \sin\beta = \cos\beta\tan\alpha - \frac{gr\cos^2\beta}{2v_0^2\cos^2\alpha} \quad \because r \neq 0$$

$$\Rightarrow \frac{gr\cos^2\beta}{2v_0^2\cos^2\alpha} = \cos\beta\tan\alpha - \sin\beta$$

$$\Rightarrow r = \frac{2v_0^2\cos^2\alpha}{g\cos^2\beta}\left[\frac{\sin\alpha\cos\beta}{\cos\alpha} - \sin\beta\right]$$

$$= \frac{2v_0^2\cos^2\alpha}{g\cos^2\beta}\left[\frac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\cos\alpha}\right]$$

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$$R = \frac{2 v_0^2 \cos\alpha}{g \cos^2\beta}\left[\sin(\alpha - \beta)\right]$$

$$R = \frac{v_0^2}{g \cos^2\beta}\left[2 \cos\alpha \sin(\alpha - \beta)\right] \longrightarrow (2)$$

$$= \frac{v_0^2}{g \cos^2\beta}\left[\sin(\alpha + \alpha - \beta) - \sin\{\alpha - (\alpha - \beta)\}\right]$$

$$= \frac{v_0^2}{g \cos^2\beta}\left[\sin(2\alpha - \beta) - \sin\beta\right]$$

$$R = \frac{v_0^2}{g \cos^2\beta}\left[\sin(2\alpha - \beta) - \sin\beta\right] \longrightarrow (3)$$

Equation (1) or (2) is used as the formula for the range on an inclined plane

The range down the plane is obtained by putting $(-\beta)$ instead of β and so

Range down the plane is

$$R = \frac{v_0^2}{g \cos^2\beta}\left[\sin(2\alpha + \beta) + \sin\beta\right] \longrightarrow (4)$$

Maximum Range

Since β is a fixed angle, therefore for a given value of v_0, the range R varies with α and will be maximum if $\sin(2\alpha - \beta)$ is max i.e

when $\sin(2\alpha - \beta) = 1$

or $2\alpha - \beta = \pi/2$

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or $2\alpha = \frac{\pi}{2} + \beta$

or $\alpha = \frac{\pi}{4} + \beta/2$

Hence max range up inclined plane is

$$R_{max} = \frac{V_0^2}{g\cos^2\beta}(1 - \sin\beta) \longrightarrow (5).$$

$$= \frac{V_0^2}{g}\frac{(1-\sin\beta)}{1-\sin^2\beta}$$

$$\boxed{R_{max} = \frac{V_0^2}{g(1+\sin\beta)}} \longrightarrow (6)$$

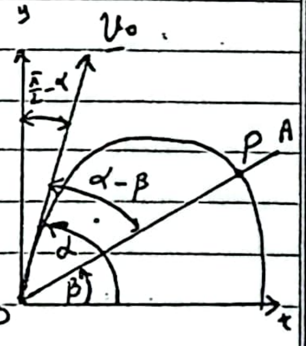
By changing β into $-\beta$, we have the maximum range down inclined plane

$$\text{Max Range down the inclined plane} = \frac{V_0^2}{g(1-\sin\beta)}$$

Also we note that for maximum range on an inclined plane, we have

$$2\alpha - \beta = \pi/2$$

$$\Rightarrow \alpha - \beta = \frac{\pi}{2} - \alpha$$



$\Rightarrow$ Angle of the direction of projection with OA i.e inclined plane = Angle of the direction of projection with vertical

Hence for max range, the direction of projection must bisect the angle between the vertical and the inclined plane.

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In Case of Max Range on inclined Plane the Focus of the Path lies in the Range

For the maximum range on an inclined plane we have

$$2\alpha - \beta = \frac{\pi}{2} \Rightarrow 2\alpha = \pi/2 + \beta$$

$$\Rightarrow \alpha - \beta = \frac{\pi}{2} - \alpha$$

Hence for max range, the direction of Projection must bisect the angle between the vertical and the inclined plane.

Now Co-ordinates of focus are

$$\left(\frac{V_0^2 \sin 2\alpha}{2g}, \frac{-V_0^2 \cos 2\alpha}{2g}\right)$$

using $2\alpha = \frac{\pi}{2} + \beta$

$$= \left(\frac{V_0^2 \sin\left(\frac{\pi}{2}+\beta\right)}{2g}, \frac{-V_0^2 \cos\left(\frac{\pi}{2}+\beta\right)}{2g}\right)$$

$$= \left(\frac{V_0^2 \cos\beta}{2g}, \frac{+V_0^2 \sin\beta}{2g}\right)$$

Now equation of line OP is

$$y = x \tan\beta$$

putting Co-Cordinates of focus in it

$$R.H.S = x\tan\beta = \frac{V_0^2 \cos\beta}{2g} \cdot \tan\beta$$

$$= \frac{V_0^2 \cos\beta}{2g} \cdot \frac{\sin\beta}{\cos\beta}$$

$$= \frac{V_0^2 \sin\beta}{2g}$$

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$$L.H.S = \frac{V_0^2 \sin\beta}{2g}$$

$\Rightarrow L.H.S = R.H.S$

$\Rightarrow$ focus lies on OP.

Thus in case of the maximum range the focus of the parabolic path lies within the range of projectile on the inclined plane

Directions of Projection for a given Range

Range on an inclined plane is

$$R = \frac{v_0^2}{g\cos^2\beta}\left[\sin(2\alpha-\beta) - \sin\beta\right]$$

$$R = \frac{v_0^2}{g\cos^2\beta}\left[\sin\{\pi-(2\alpha-\beta)\} - \sin\beta\right]$$

Thus for a given velocity of projection when the value of the range, other than the max is given $\sin(2\alpha-\beta)$ becomes known and we have from above two expressions that we get two values of $2\alpha-\beta$ each less than 180° and their sum is 180°.

If α_1, α_2 be corresponding values of α for these two angles $2\alpha-\beta$, then

$$2\alpha_1-\beta+2\alpha_2-\beta = \pi$$

$$\Rightarrow 2\alpha_1-\beta = \pi-(2\alpha_2-\beta)$$

$$\Rightarrow 2\alpha_1-\beta = \frac{\pi}{2}+\frac{\pi}{2}-(2\alpha_2-\beta)$$

$$\Rightarrow 2\alpha_1-\frac{\pi}{2}-\beta = \frac{\pi}{2}+\beta-2\alpha_2$$

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$$\Rightarrow \alpha_1 - \frac{1}{2}\left(\frac{\pi}{2}+\beta\right) = \frac{1}{2}\left(\frac{\pi}{2}+\beta\right) - \alpha_2$$

$\frac{1}{2}\left(\frac{\pi}{2}+\beta\right)$ being the direction for the maximum range, it follows that for a given range there are two directions of projection which are equally inclined to the direction for max. range.

Time of Flight on inclined Plane

The time taken by the projectile during its motion from O to P is called the time of flight. If the particle strikes the plane after a time T, Then

$$OL = R\cos\beta$$

is the horizontal distance covered by the projectile during time T.

$$\text{Horizontal distance} = (V_0\cos\alpha)T$$

$$R\cos\beta = V_0\cos\alpha\, T$$

$$T = \frac{R\cos\beta}{V_0\cos\alpha}$$

But $R = \frac{2v_0^2}{g\cos^2\beta}\sin(\alpha-\beta)\cos\alpha$

using This we have

$$\text{Time of flight} = T = \frac{2V_0\sin(\alpha-\beta)}{g\cos\beta}$$

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Velocity at Point P

We have already proved that velocity v of projectile at any height y from the projection level is given by

$$v^2 = v_0^2 - 2gy$$

Height of point P above projection level is $r \sin\beta$.

Therefore velocity v of the projectile at point P is given by

$$v^2 = v_0^2 - 2g(r \sin\beta)$$

Direction at Point P

Let θ be the angle which the velocity (direction) at P makes with the horizontal. Then

$$\tan\theta = \frac{\text{Vertical velocity at P}}{\text{Horizontal velocity at P}}$$

$$= \frac{V_0 \sin\alpha - gT}{V_0 \cos\alpha}$$

where $T = \dfrac{2V_0 \sin(\alpha - \beta)}{g \cos\beta}$

Therefore

$$\tan\theta = \frac{V_0 \sin\alpha}{V_0 \cos\alpha} - \frac{g}{V_0 \cos\alpha} \times \frac{2V_0 \sin(\alpha-\beta)}{g \cos\beta}$$

$$= \tan\alpha - 2\frac{\sin(\alpha-\beta)}{\cos\alpha \cos\beta}$$

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$$\tan\theta = \tan\alpha - 2\left[\frac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\cos\alpha\cos\beta}\right]$$

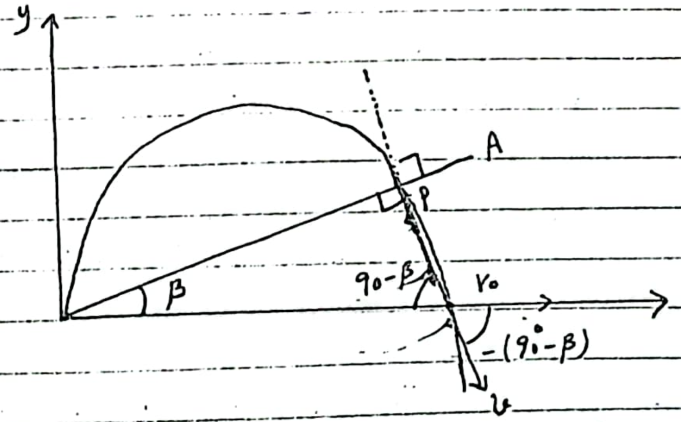
$$= \tan\alpha - 2(\tan\alpha - \tan\beta)$$

$$\tan\theta = 2\tan\beta - \tan\alpha$$

Thus direction at point P is given by

$$\boxed{\tan\theta = 2\tan\beta - \tan\alpha}$$

The Condition that the Particle may strike the Plane at right angle



When the projectile strikes the plane at P at right angle, the direction of motion of the particle at P is downward making an angle $\frac{\pi}{2}-\beta$ with the horizontal in -ve sense as shown. so in the expression

$$\tan\theta = 2\tan\beta - \tan\alpha$$

we put $\theta = -(\pi/2 - \beta)$

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$$\tan(\beta - \tfrac{\pi}{2}) = 2\tan\beta - \tan\alpha$$

$$-\tan(\tfrac{\pi}{2} - \beta) = 2\tan\beta - \tan\alpha$$

$$\cot\beta = \tan\alpha - 2\tan\beta$$

Adding $\tan\alpha$ to both sides, we have

$$\frac{1}{\tan\beta} + \tan\alpha = 2(\tan\alpha - \tan\beta)$$

$$\frac{1 + \tan\alpha\tan\beta}{\tan\beta} = 2(\tan\alpha - \tan\beta)$$

$$\frac{1}{\tan\beta} = 2\,\frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

$$\cot\beta = 2\tan(\alpha - \beta)$$

Thus required Condition that the particle may strike the plane at right angle is

$$\boxed{\cot\beta = 2\tan(\alpha - \beta)}$$

Note# In the above article the direction of projection is expressed in terms of the angle with the horizontal. It can also be expressed in terms of the angle with the inclined plane. In working problem care must be taken to see which of these angles is given.

Now we express the above formulas in terms of angle of projection with inclined plane.

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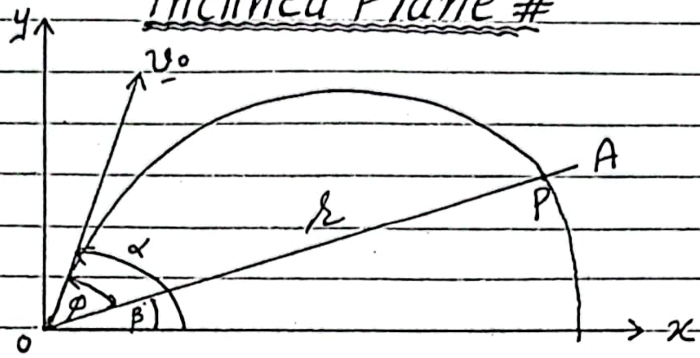
Formulas for Projection above inclined Plane in terms of angle of Projection with inclined Plane



(a) # Range on the inclined plane is given by

$$R = \frac{2 V_0^2 \cos\alpha \sin(\alpha-\beta)}{g \cos^2\beta}$$

Here β is the inclination of plane and α is the angle of projection of projectile with the horizontal.

Let $\varphi = \alpha - \beta$ angle of projection with inclined plane

Then $\alpha = \varphi + \beta$

and range up inclined plane becomes

$$R = \frac{2 V_0^2 \sin\varphi \cos(\varphi+\beta)}{g \cos^2\beta}$$

(b) # Max range on inclined plane becomes

$$R_{max} = \frac{v_0^2}{g\cos^2\beta}\left[1-\sin\beta\right]$$

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Range down the plane is

$$R = \frac{2V_0^2 \cos\alpha \sin(\alpha+\beta)}{g\cos^2\beta}$$

Now $\alpha - \beta = \varphi$

changing β into $-\beta$

$\therefore \alpha + \beta = \varphi$

$\Rightarrow \alpha = \varphi - \beta$

Range down the plane becomes

$$R = \frac{2V_0^2 \sin\varphi \cos(\varphi-\beta)}{g^2\cos^2\beta}$$

Max range down the plane remain same and is given by

$$R_{max} = \frac{v_0}{g\cos^2\beta}\left[1+\sin\beta\right]$$

(C) Times of flight up and down inclined plane are given by

Time of flight up inclined plane $= T = \frac{2V_0 \sin(\alpha-\beta)}{g\cos\beta}$

Tim of flight down inclined plane $= T = \frac{2V_0 \sin(\alpha+\beta)}{g\cos\beta}$

Now for up inclined plane $\alpha - \beta = \varphi$

and for dow inclined plane $\alpha + \beta = \varphi$

Time of flight up and down inclined plane is

$$T = \frac{2V_0 \sin\varphi}{g\cos\beta}$$

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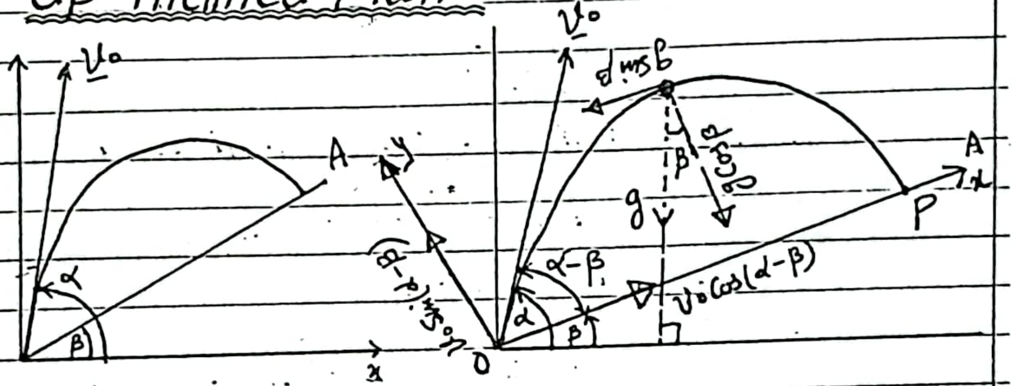
Alternate Method To Find Range and Time of Flight on an inclined Plane

When a particle is projected from point O on an inclined plane of inclination β, in a vertical plane through the inclined plane with velocity v_0 making angle α with the horizontal, its range on the plane and the time of flight can also be obtained by resolving its velocity v_0 and acceleration g due to gravity along and perpendicular to the plane i.e taking x-axis along the line of the greatest slope OA and y-axis perpendicular to OA (i.e inclined plane

(a) Up Inclined Plane



The direction of projection makes angle $\alpha-\beta$ with the inclined plane. The resolved parts of the initial velocity and of acc g along and perpendicular to OP are

	Along OP	Perpendicular to OP
Velocity	$v_0 \cos(\alpha-\beta)$	$v_0 \sin(\alpha-\beta)$
Acceleration	$-g \sin\beta$	$-g \cos\beta$

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Motion Perpendicular to OP

Initial Vel $= v_0 \sin(\alpha - \beta)$

acc $= -g\cos\beta$

Velocity at any time $t = v_y = \dot{y} = ?$

$$v = u + at$$

$$\dot{y} = v_y = V_0 \sin(\alpha - \beta) - gt\cos\beta$$

$$\boxed{v_y = V_0 \sin(\alpha - \beta) - gt\cos\beta} \longrightarrow (1)$$

Distance perpendicular to plane at any tim t is

$$\boxed{y = V_0 \sin(\alpha - \beta)t - \frac{1}{2}gt^2\cos\beta} \longrightarrow (2)$$

OR

We can derive above equations as

$$acc = \ddot{y} = -g\cos\beta$$

Integrating w.r.t t

$$\dot{y} = -g\cos\beta\, t + A \longrightarrow (3)$$

at $t = 0$ $\quad \dot{y} = v_y = V_0 \sin(\alpha - \beta)$

$$\Rightarrow \quad V_0 \sin(\alpha - \beta) = 0 + A$$

Using this value of A in (3), we get

$$\dot{y} = V_0 \sin(\alpha - \beta) - gt\cos\beta$$

Again integrating w.r.t. t

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$$y = V_0 \sin(\alpha-\beta).t - \frac{1}{2} g t^2 \cos\beta + B \longrightarrow (4)$$

at $t=0$ $y=0$

$\Rightarrow 0 = 0 + 0 + B$

putting this value of B in (4), we get

$$y = V_0 \sin(\alpha-\beta)t - \frac{1}{2} g t^2 \cos\beta \longrightarrow (5)$$

Motion along Plane (OP)

Initial Vel $= V_0 \cos(\alpha-\beta)$

$Acc = \ddot{x} = -g \sin\beta$

Velocity at any time $t = V_x = \dot{x} = ?$

$$v = u + at$$

$$\dot{x} = v_x = V_0 \cos(\alpha-\beta) - g t \sin\beta$$

$$\boxed{\dot{x} = v_x = V_0 \cos(\alpha-\beta) - g t \sin\beta} \longrightarrow (6)$$

Distance Covered along plane at any tim t is

$$\boxed{x = V_0 \cos(\alpha-\beta)t - \frac{1}{2} g t^2 \sin\beta} \longrightarrow (7)$$

OR

Equations (6) & (7) can be derived as

$acc = \ddot{x} = -g \sin\beta$

Integrating w.r.t t

$$v_x = \dot{x} = -g \sin\beta . t + C \longrightarrow (8)$$

at $t=0$ $v_x = V_0 \cos(\alpha-\beta)$

$\Rightarrow V_0 \cos(\alpha-\beta) = C$

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Putting this value of c in ⑧

$$v_x = \dot{x} = V_0 \cos(\alpha-\beta) - gt\sin\beta \longrightarrow (9)$$

Again integrating w.r.t t

$$x = V_0 \cos(\alpha-\beta).t - \frac{1}{2}gt^2\sin\beta + D \longrightarrow (10)$$

at $t = 0$ $x = 0$

$\Rightarrow 0 = D$

Putting this value of D in ⑩, we get

$$x = V_0 \cos(\alpha-\beta)t - \frac{1}{2}gt^2\sin\beta \longrightarrow (11)$$

Time of flight up inclined Plane

The time of flight is the value of t when $y = 0$ (when particle strikes the plane at P).

Putting $y = 0$ in ②

$$V_0 \sin(\alpha-\beta)t - \frac{1}{2}gt^2\cos\beta = 0$$

$$\Rightarrow t\left(V_0 \sin(\alpha-\beta) - \frac{1}{2}gt\cos\beta\right) = 0$$

$$\Rightarrow t = 0 \quad \text{or} \quad V_0 \sin(\alpha-\beta) - \frac{1}{2}gt\cos\beta = 0$$

$$\text{or} \quad t = \frac{2V_0 \sin(\alpha-\beta)}{g\cos\beta}$$

Thus time of flight up inclined plane is

$$T = \frac{2V_0 \sin(\alpha-\beta)}{g\cos\beta}$$

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Range

The range (R) up the plane is the distance OP i.e the value of distance x along the plane when

$$t = \frac{2V_0 \sin(\alpha-\beta)}{g\cos\beta}$$

Putting this value of t in (7)

$$R = x = V_0\cos(\alpha-\beta)\left(\frac{2V_0\sin(\alpha-\beta)}{g\cos\beta}\right) - \frac{1}{2}g\sin\beta\left(\frac{2V_0\sin(\alpha-\beta)}{g\cos\beta}\right)^2$$

$$= \frac{2V_0^2\sin(\alpha-\beta)\cos(\alpha-\beta)}{g\cos\beta} - \frac{2V_0^2\sin^2(\alpha-\beta)\sin\beta}{g\cos^2\beta}$$

$$= \frac{2V_0^2\sin(\alpha-\beta)}{g\cos^2\beta}\left[\cos(\alpha-\beta)\cos\beta - \sin(\alpha-\beta)\sin\beta\right]$$

$$= \frac{2V_0^2\sin(\alpha-\beta)}{g\cos^2\beta}.\cos(\alpha-\beta+\beta)$$

$$R = \frac{2V_0^2\sin(\alpha-\beta)\cos\alpha}{g\cos^2\beta}$$

$$\boxed{R = \frac{2V_0^2\sin(\alpha-\beta)\cos\alpha}{g.\cos^2\beta}}$$

★ Sincerity to the parents, trust in God and Continuous struggle always make a man to reach his destination.

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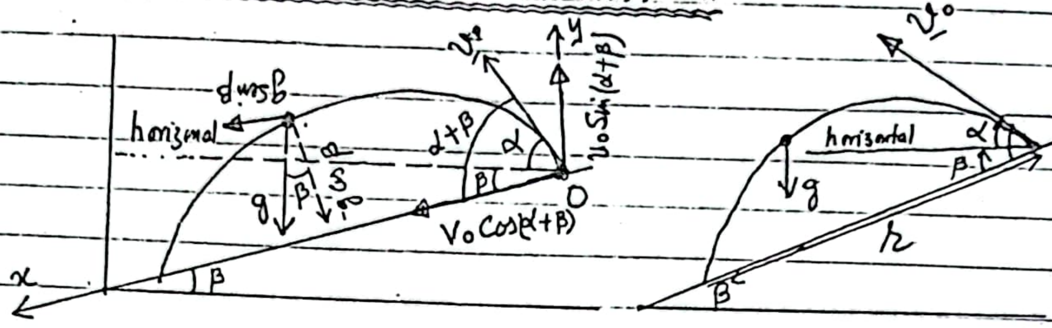
(b) Down Inclined Plane



If a particle is projected down the plane with initial velocity v_0 making angle α with the horizontal. Then taking axes as shown.

The direction of projection makes angle $\alpha+\beta$ with the inclined plane. The resolved parts of the initial velocity v_0 and of acc g along and perpendicular to the plane are

	Along plane	Perpendicular to plane
Velocity	$V_0 \cos(\alpha+\beta)$	$V_0 \sin(\alpha+\beta)$
Acceleration	$g \sin\beta$	$-g\cos\beta$

(a) Motion Perpendicular to Plane

Initial Vel $= V_0 \sin(\alpha+\beta)$

acc $= -g\cos\beta$

Velocity at any time $t = v_y = \dot{y} = ?$

$$v = u + at$$

$$\dot{y} = v_y = V_0 \sin(\alpha+\beta) - gt\cos\beta$$

$$\boxed{\dot{y} = v_y = V_0 \sin(\alpha+\beta) - gt\cos\beta} \rightarrow (12)$$

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Distance perpendicular to plane at any time t is

$$y = V_0 \sin(\alpha+\beta) t - \frac{1}{2} g t^2 \cos\beta \longrightarrow (13)$$

OR

$$acc = \ddot{y} = -g\cos\beta$$

Integrating w.r.t time

$$v_y = \dot{y} = -g\cos\beta t + E \longrightarrow (14)$$

at $t=0$ $\quad v_y = V_0 \sin(\alpha+\beta)$

$\Rightarrow$

$$V_0 \sin(\alpha+\beta) = 0 + E$$

putting this value of E in (14), we get

$$V_y = \dot{y} = V_0 \sin(\alpha+\beta) - g t \cos\beta$$

Again integrating w.r.t time:

$$y = V_0 \sin(\alpha+\beta) t - \frac{1}{2} g t^2 \cos\beta + F \rightarrow (15)$$

at $t=0$ $\quad y=0$

$\Rightarrow \quad F=0$

putting this value of F in (15)

$$y = V_0 \sin(\alpha+\beta) t - \frac{1}{2} g t^2 \cos\beta \longrightarrow (16)$$

Motion along the Plane

Initial vel $= V_0 \cos(\alpha+\beta)$

$acc = g \sin\beta$

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Velocity at any time $t = V_x = \dot{x} = ?$

$$v = u + at$$

~~$\dot{x} = V_x = V_0 \cos(\alpha \mp \beta) + gt\sin\beta$~~

$$\dot{x} = V_x = V_0 \cos(\alpha + \beta) + gt\sin\beta \longrightarrow (17)$$

Distance covered along the plane at any time t is given by

$$x = ut + \frac{1}{2}at^2$$

$$x = V_0 \cos(\alpha + \beta)t + \frac{1}{2}gt^2\sin\beta \longrightarrow (18)$$

Time of Flight Down inclined Plane

The time of flight is the value of t when $y = 0$ i.e particle strikes the inclined plane

putting $y = 0$ in (13)

$$0 = V_0 \sin(\alpha + \beta)t - \frac{1}{2}gt^2\cos\beta$$

$$\Rightarrow t\left[V_0 \sin(\alpha + \beta) - \frac{1}{2}gt\cos\beta\right] = 0$$

$$\Rightarrow t = 0 \quad \text{or} \quad V_0 \sin(\alpha + \beta) - \frac{1}{2}gt\cos\beta = 0$$

$$\text{or} \quad t = \frac{2V_0 \sin(\alpha + \beta)}{g\cos\beta} \longrightarrow (19)$$

Range down inclined Plane

Putting value of t from (19)

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in (18) range ℓ is

$$\ell = x = V_0 \cos(\alpha+\beta) \cdot \frac{2V_0 \sin(\alpha+\beta)}{g\cos\beta} + \frac{1}{2} g \sin\beta \left[\frac{2V_0 \sin(\alpha+\beta)}{g\cos\beta}\right]^2$$

$$\ell = \frac{2V_0^2 \sin(\alpha+\beta)\cos(\alpha+\beta)}{g\cos\beta} + \frac{2V_0^2 \sin^2(\alpha+\beta)\sin\beta}{g\cos^2\beta}$$

$$= \frac{2V_0^2 \sin(\alpha+\beta)}{g\cos^2\beta}\left[\cos(\alpha+\beta)\cos\beta + \sin(\alpha+\beta)\sin\beta\right]$$

$$= \frac{2V_0^2 \sin(\alpha+\beta)\cos(\alpha+\beta-\beta)}{g\cos^2\beta}$$

$$\boxed{\ell = \frac{2V_0^2 \sin(\alpha+\beta)\cos\alpha}{g\cos^2\beta}}$$

Condition to strike the Plane at Right angle

If the particle strikes the plane perpendicularly then its velocity at that instant parallel to the plane is zero.

Now for motion up inclined plane velocity at any time t along the plane is given by

$$\dot{x} = V_x = V_0 \cos(\alpha-\beta) - g t \sin\beta$$

and Time of flight $= T = \dfrac{2V_0 \sin(\alpha-\beta)}{g\cos\beta}$

Now if the particle strikes inclined plan perpendicularly after time T, we have

$$V_x (\text{at } T) = 0$$

$$\Rightarrow V_0 \cos(\alpha-\beta) - g T \sin\beta = 0$$

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$$\Rightarrow \frac{V_o \cos(\alpha-\beta)}{g \sin\beta} = T$$

putting value of $T = \frac{2V_o \sin(\alpha-\beta)}{g\cos\beta}$

we have

$$\frac{V_o \cos(\alpha-\beta)}{g \sin\beta} = \frac{2V_o \sin(\alpha-\beta)}{g\cos\beta}$$

$$\frac{\cos\beta}{\sin\beta} = 2\frac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)}$$

$$\cot\beta = 2\tan(\alpha-\beta)$$

which is the required condition

Comparison of Some Formulas of Projectile for flight w.r.t Horizontal level & w.r.t level of an inclined Plane of inclination β

Note # for up inclined motion of projectile projected with vel v_o making angle α with the horizontal, initial horizontal and vertical velocities are $V_o\cos\alpha$, $V_o\sin\alpha$ respectively while initial velocities ||al and ⊥ar to the inclined plane are $V_o\cos(\alpha-\beta)$; $V_o\sin(\alpha-\beta)$. Components of acc. g along and ⊥ar to the plane are $g\sin\beta$, $g\cos\beta$. Also angle of projection with inclined plane will be $\alpha-\beta$.

Horizontal level	Level of the inclined Plane
Horizontal & vertical velocities at any time t are	Velocities along and ⊥ar to the plane at any time t are
$V_x = V_o\cos\alpha \rightarrow (1)$	$V_{\parallel} = V_o\cos(\alpha-\beta) - (g\sin\beta)t \rightarrow (1')$
$V_y = V_o\sin\alpha - gt \rightarrow (2)$	$V_{\perp} = V_o\sin(\alpha-\beta) - (g\cos\beta)t \rightarrow (2')$

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(b) Horizontal & Vertical distances at any time t are (To memorize integrate ① & ② w.r.t t without constant of integration	Distances parallel & perp. to the plane at any time t. (To memorize integral ①' & ②' w.r.t t without constant of integration
$x = V_0 \cos\alpha\, t \longrightarrow ③$	$d_{\parallel} = V_0 \cos(\alpha-\beta)t - \frac{1}{2}gt^2 \sin\beta \longrightarrow ③'$
$y = V_0 \sin\alpha\, t - \frac{1}{2}gt^2 \longrightarrow ④$	$d_{\perp} = V_0 \sin(\alpha-\beta)t - \frac{1}{2}gt^2 \cos\beta \longrightarrow ④'$
(c) Time of flight over Horizontal level is	Time of flight over the inclined plane
$T = \frac{2V_0 \sin\alpha}{g} \longrightarrow ⑤$ (α → angle of projection with horizontal; g → acc ⊥ar to horizontal)	$T = \frac{2V_0 \sin(\alpha-\beta)}{g\cos\beta} \longrightarrow ⑤'$ (α−β → angle of projection with the plane; g cos β → acc ⊥ar to the plane)
(C)# Horizontal range	Range on an inclined plane.
$R = \frac{2V_0^2 \sin\alpha \cos\alpha}{g} \longrightarrow ⑥$	$r = \frac{2V_0^2 \sin(\alpha-\beta)\cdot\cos\alpha}{g\cos^2\beta} \longrightarrow ⑥'$
Here Multiply T by $V_0\cos\alpha$ to get R	Here Multiply T by $\frac{V_0\cos\alpha}{\cos\beta}$ to get r
Maximum Horizontal range is	Maximum range up inclined plane is
$R_{max} = \frac{V_0^2}{g}$	$r_{max} = \frac{V_0^2}{g\cos^2\beta}(1-\sin\beta)$

Note # To change the above all formulas for up inclined plane into formulas for down inclined plane change β into −β in all of these formulas.

★ By Muhammad Hussain Lecturer (Maths) Govt College Asghar Mall Rawalpindi #

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Revision from Calculus

Locus # When a point moves under a given condition or conditions, then the path traced by the point called the locus of the point

e.g. Let C be a fixed point and suppose a point P moves under the condition that its distance from C should always be r. Then P describes a circle with centre C and radius r.
∴ Hence the locus of the moving point P is the circle of radius r and centre C

Equation to the Locus

When a (particle) point moves under a given condition its abscissa (x-co-ordinate) are ordinate (y-co-ordinate) are connected together by a definite relation (or equation). This relation is called the equation to the locus of the moving point. Since the abscissa and ordinate of the moving point are always denoted by x and y respectively. Hence we have the following definition

The equation to the locus is the relation between x & y which is satisfied by the co-ordinates of every point on the locus and which is not satisfied by the co-ordinates of any point outside the locus. This equation should not contain any unknown quantity.

Conic # Any curve which is the locus of a point which moves so that the ratio of its distance from a fixed point to its distance from a fixed

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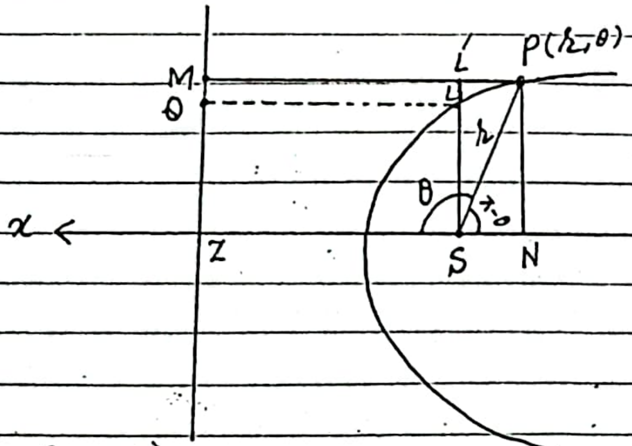
line is Constant. The ratio is the eccentricity (اختلاف) of the curve, the fixed point is the focus, and the fixed line is the directrix. The eccentricity is always denoted by e

When $e = 1$, the Conic is a parabola
$e < 1$ the Conic is an ellipse
$e > 1$. the Conic is a Hyperbola

These are called Conics or Conic sections since they can always be obtained by taking plane sections of a Conical surface.

Degenerate Conic # A point, a straight line, or a pair of straight lines, which is a limiting form of a Conic

Polar Equation of Conic



Let $\overrightarrow{SZ}$ be the axis; S the focus and ZM be the directrix of the Conic. Let l be the semi-latus rectum SL. Draw ⊥ar LQ to ZM. Then by definition of Conic

$$\therefore \frac{SL}{SZ} = \frac{SL}{LQ} = e$$

$$\Rightarrow SL = eLQ = eSZ$$

$$SZ = \frac{SL}{e} = l/e \rightarrow ①$$

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Taking SZ as the initial line and focus S as pole (origin) let $P(r,\theta)$ be any point on the conic. Draw PN ⊥ar to SZ and PM ⊥ar to ZM. Then by definition of conic

$$\frac{SP}{PM} = e$$

$$\Rightarrow SP = e\,PM$$
$$= e(PL' + L'M)$$
$$= e(SN + SZ)$$

$$\Rightarrow r = e\left[\frac{l}{e} + r\cos(\pi-\theta)\right] \qquad SN = r\cos(\pi-\theta),\ SZ = l/e$$

$$= e\left[\frac{l}{e} - r\cos\theta\right]$$

$$= l - er\cos\theta$$

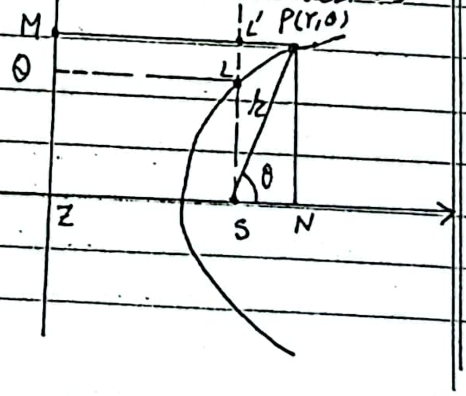
$$r + er\cos\theta = l$$
$$r(1+e\cos\theta) = l$$
$$1 + e\cos\theta = \frac{l}{r}$$
$$\frac{l}{r} = 1 + e\cos\theta \rightarrow (1)$$

This is the required equation.

Note# (1) Note that θ is measured in clockwise sense and initial line is taken along −ve x-axis (axis). If θ is measured counterclockwise and +ve x-axis is taken as initial line (axis), then we have

$$SP = e(SZ + SN)$$
$$r = e\left(\frac{l}{e} + r\cos\theta\right)$$
$$= e\frac{l}{e} + er\cos\theta$$
$$\frac{l}{r} = 1 - e\cos\theta \rightarrow (2)$$

Here $\vec{ZS}$ is axis



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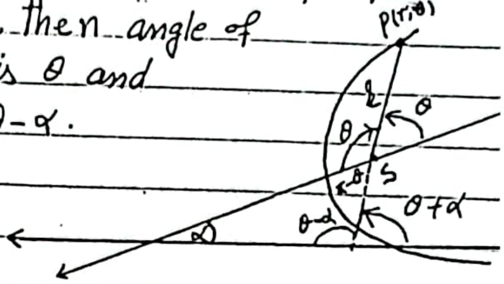
2) # We have actually measured θ w.r.t axis of the conic which in both cases is along the initial line. If in case of equation ① the axis of of the con is inclined at an angle α, then angle of SP = r w.r.t axis of conic is θ and w.r.t to x-axis clockwise is θ − α.



So equation of conic ① is

$$\frac{l}{r} = 1 + e\cos(\theta - \alpha)$$

If θ is measured counterclockwise, then angle of SP = r with initial line counterclockwise is θ + α and equation of conic is

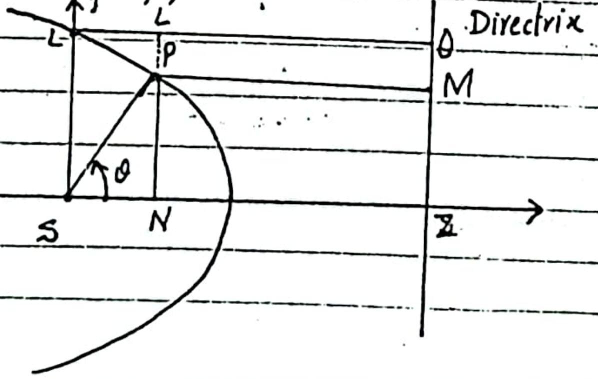
$$\frac{l}{r} = 1 - e\cos(\theta + \alpha)$$

3) # When θ is measured from the vertex, the equation of conic with focus as pole is always of the form

$$\frac{l}{r} = 1 + e\cos\theta$$

Different forms of Polar Equation of Conic with focus as Pole

When angle is measured from x-axis and focus is taken at origin the equations of conic in polar form change depending upon the location of directrix as.



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$\frac{SL}{LQ} = e \Rightarrow LQ = \frac{SL}{e} = \frac{l}{e}$

$\Rightarrow SZ = \frac{l}{e}$

$PM = \frac{SP}{e}$

$\Rightarrow SP = e\,PM$

$= e\,L'Q$

$r = e(LQ - LL')$

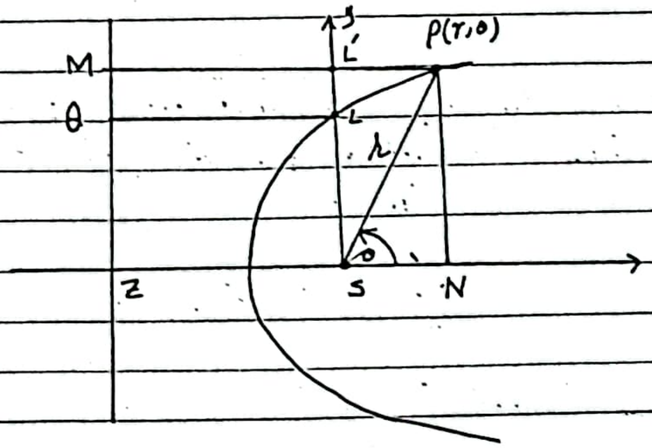
$= e\left(\frac{l}{e} - SN\right)$

$= e\left(\frac{l}{e} - r\cos\theta\right)$

$\Rightarrow r(1 + e\cos\theta) = l$

$\Rightarrow r = \frac{l}{1 + e\cos\theta}$

2) #



$SZ = LQ$

$\frac{LS}{LQ} = e$

$\frac{l}{e} = LQ = SZ$

$SP = PM\,e$

$= e(ML' + L'P)$

$= e(ZS + SN)$

$r = e\left(\frac{l}{e} + r\cos\theta\right)$

$r(1 - e\cos\theta) = l$

$r = \frac{l}{1 - e\cos\theta}$

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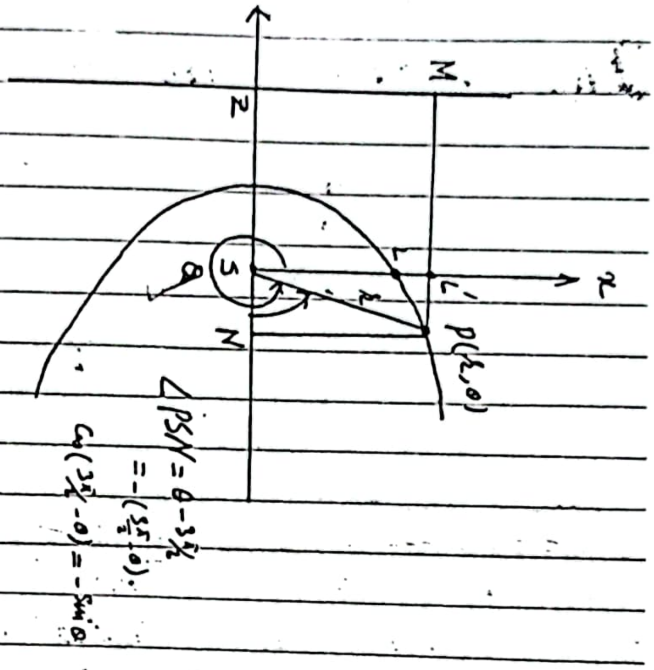
3)



$\therefore \frac{l}{e} = SZ$

$PS = PM e = e(SZ + SN)$

$r = e\left(\frac{l}{e} + r\cos\left(\frac{3\pi}{2} - \theta\right)\right)$

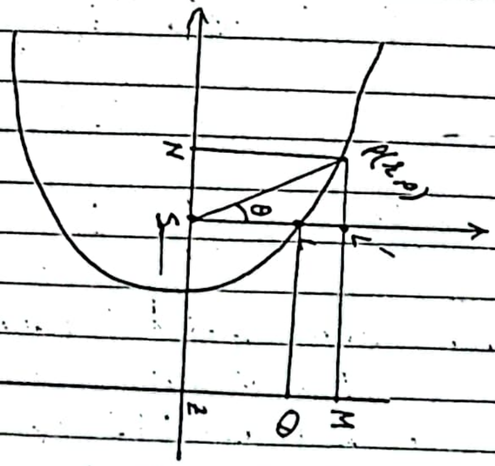
$= l - er\sin\theta$

$r(1 + e\sin\theta) = l$

$r = \frac{l}{1 + e\sin\theta}$

4) #

$\frac{l}{e} = SZ$



$ePM = SP$

$SP = e(SZ + SN)$

$r = e\left(\frac{l}{e} + r\sin\theta\right)$

$= l + er\sin\theta$

$r(1 - e\sin\theta) = l$

$r = \frac{l}{1 - e\sin\theta}$

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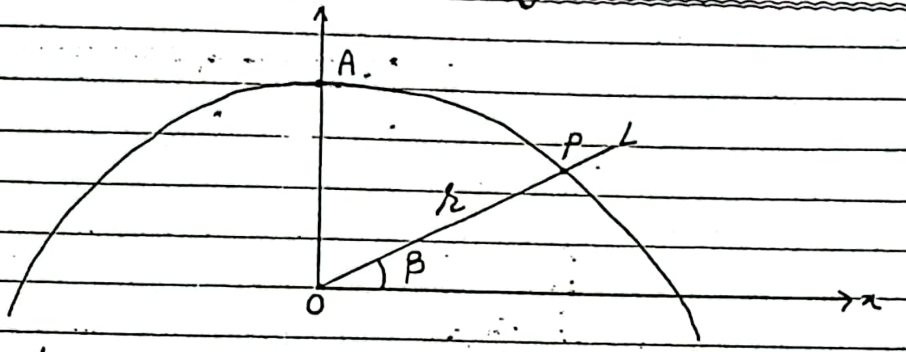
Outer Boundry Reached by Projectiles Projected with given Same Velocity



When a projectile is projecticle is projecte from a point O on inclined plane of inclination β with initial velocity v_0 making angle α with horizontal level ($\alpha > \beta$). Then maximum range on the inclined plane is given by

$$r = \frac{v_0^2}{g(1+\sin\beta)} \rightarrow (1)$$

for given velocity v_0 and β when

$$2\alpha - \beta = \frac{\pi}{2} \rightarrow (2)$$

from (1)

$$\frac{1}{r} = \frac{g}{v_0^2}\left(\frac{1}{1+\sin\beta}\right) = \frac{g}{v_0^2}(1+\sin\beta)$$

$$\Rightarrow \frac{v_0^2}{gr} = 1+\sin\beta$$

$$\Rightarrow \frac{\left(\frac{v_0^2}{g}\right)}{r} = 1+\sin\beta$$

which polar equation of parabola ($e=1$) with focus at origin O and semi-latus-rectum $\frac{v_0^2}{g}$, axis vertical and vertex A upwards. Since OP, here is maximum range provided direction of OP i.e β is such that $2\alpha - \beta = \frac{\pi}{2}$. Therefore any radius drawn from O will give the max range in direction $\beta = 2\alpha - \frac{\pi}{2}$

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projected from o with velocity V_0. In other words if a particle is projected from o with angle of projection $\alpha = \frac{\pi}{4} - \frac{\beta}{2}$ where β is inclination of an imaginary inclined plane through o, then particle will reach the parabola

$$\frac{\left(\frac{v_0^2}{g}\right)}{r} = 1 + \sin\beta$$

and will not cross the parabola. This parabola is an outer boundary to the region that can be reache by projectiles from o with given velocity v_0

One Parameter Family of Curves &

Envelope

An equation of the form

$$f(x, y, \alpha) = 0$$

which gives a curve corresponding to each particu value of α represents a family of curves for different values of α. The variable α, which is different for different curves and constant for one curve is called parameter for the family. e.g

$$(x-a)^2 + y^2 = 4$$

represents a circle for different values of a. all members have common property that their radius is 2 and their centres lie on x-axis.

Envelope of one Parameter family of Curves

The envelope of a one-parameter family of curves is a curve that tangent to (has common tangent with) every curve of the family or which touches every curve of the family.

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Method to find envelope

Consider one parameter family of curves

$$f(x, y, \alpha) = 0 \longrightarrow (1)$$

where α is parameter
put partial derivative of f w.r.t. α equal to zero i.e

$$\frac{\partial f}{\partial \alpha} = 0 \longrightarrow (2)$$

Eliminating α from (1) & (2) we get the envelope of the family of curves (1).

OR

If one parameter family of curve is quadratic in parameter say α i.e is of the form

$$A\alpha^2 + B\alpha + C = 0.$$

Then the envelope is obtained by equating disc zero i.e envelope is

$$B^2 - 4AC = 0$$

$$\text{or} \quad B^2 = 4AC$$

Parabola of Safety

in vertical plane through pt of pro: The path of projectile projected with initial vel v_0 making angle α with horizontal is given by

$$y = x\tan\alpha - \frac{gx^2}{2v_0^2\cos^2\alpha}$$

For a given value of v_0, this trajectory depends on α and give different trajectories for different values α. Now each trajectory passes through the point of projection and has a point which is at max. distance from the point of projection. So we define parabola of safety as:

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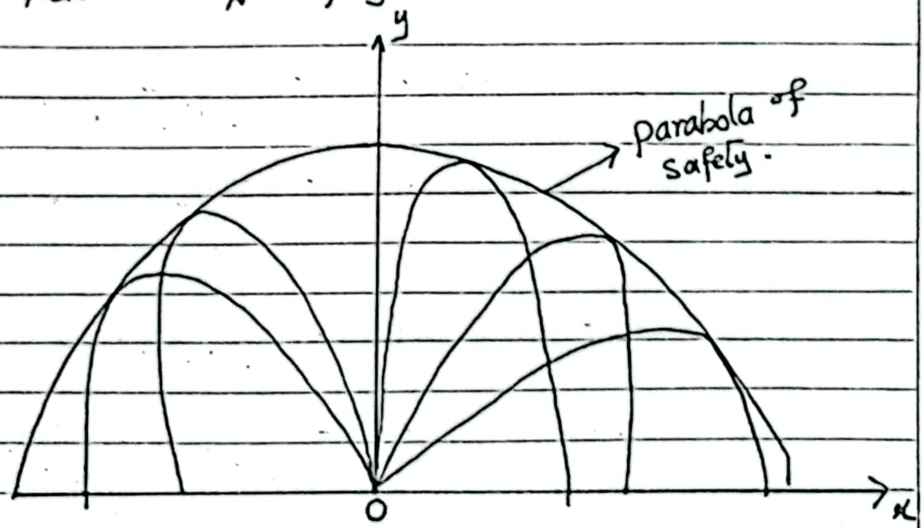
Def # A parabolic curve in vertical plane through the point of projection, which has focus at the point of projection, ~~(encloses)~~ surrounds all trajectories of a projectile projected with same velocity but different angles of projection and touches each trajectory at that point which lies at maximum distance from the point of projection.

OR

When a projectile or projectiles are projected with same velocity of projection but with different angles of projection we get different trajectories through the point of projection, then a curve, in plane of projection which envelopes or surrounds all these trajectories and touches all these trajectories at their points which are at maximum distance from the point of projection is a parabola with focus at the point of projection and is called Parabola of Safety.



Remarks # As the parabola of safety touches each trajectory of projectile at a point which is at maximum distance from the point of projection and is reached by the projectile with a given velocity u_0 for some particular angle of projection, therefore the parabola of

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is a boundry curve which contains all points in a vertical plane; which can be reached by a projectile with velocity v_0 or all projectiles with velocity of projection v_0. Now if a gun at O has muzzle velocity v_0, in any direction, then a man outside the parabola of safety remains safe because fire with vel v_0, in any direction will not cross the parabola of safety until its muzzle velocity v_0 is changed. Due to this very reason we name this parabolas as parabola of safety. And since it covers (envelopes) all the parabolas (trajectories), therefore it is also called enveloping parabola or simply the envelope of the family of the parabolas.

Equation of the Parabola of Safety:

The equation of the trajectory of a projectile, projected from O, in a vertical plane, with velocity v_0 making angle α with the horizontal levels is given by

$$y = x\tan\alpha - \frac{gx^2}{2v_0^2\cos^2\alpha}$$

$$= x\tan\alpha - \frac{gx^2}{2v_0^2}\sec^2\alpha$$

$$= x\tan\alpha - \frac{gx^2}{2v_0^2}(1+\tan^2\alpha)$$

$$= x\tan\alpha - \frac{gx^2}{2v_0^2} - \frac{gx^2}{2v_0^2}\tan^2\alpha$$

or $$\frac{gx^2}{2v_0^2}\tan^2\alpha - x\tan\alpha + \left(y + \frac{gx^2}{2v_0^2}\right) = 0 \longrightarrow (1)$$

For a given value of v_0, this trajectory depends upon α and give a family of curves for different values of α. Therefore (1) represents one-parameter family of curves and is quadratic in $\tan\alpha$ (i.e. α).

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Since the parabola of safety touches all the members of this family, therefore it is envelope of this family and its equation is given by

Disc. of ① = 0

$$\Rightarrow x^2 - 4\left(\frac{gx^2}{2V_0^2}\right)\left(y + \frac{gx^2}{2V_0^2}\right) = 0$$

$$\Rightarrow 1 - 2\left(\frac{g}{V_0^2}\right)\left(y + \frac{gx^2}{2V_0^2}\right) = 0 \quad \because x \neq 0 \text{ dividing by } x^2$$

$$\Rightarrow 1 - \frac{2g}{V_0^2}y - \frac{g^2x^2}{V_0^4} = 0$$

$$\frac{g^2x^2}{V_0^4} = -\frac{2gy}{V_0^2} + 1$$

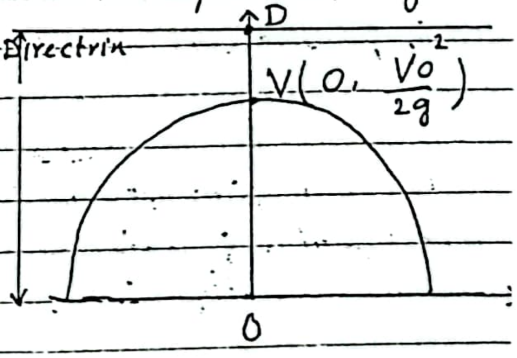
$$\Rightarrow x^2 = -\frac{2V_0^2}{g}y + \frac{V_0^4}{g^2}$$

$$\boxed{x^2 = -\frac{2V_0^2}{g}\left(y - \frac{V_0^2}{2g}\right)}$$

which is the required equation of parabola of safety with vertex

at $\left(0, \frac{V_0^2}{2g}\right)$, Latus rectum $= \frac{2V_0^2}{g}$

and it faces downward in vertical plane through the point of projection.



Focus

x-co-ordinate of the focus = x-co-ordinate of the vertex = 0

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y-Co-ordinate of focus = y-co-ordinate of vertex − $\frac{1}{4}$ (Latus-Rectum)

$$= \frac{V_0^2}{2g} - \frac{1}{4}\left(\frac{2V_0^2}{g}\right)$$

$$= \frac{V_0^2}{2g} - \frac{V_0^2}{2g} = 0$$

Hence focus = F(0,0)
i.e the focus of parabola of safety is at origin.

Equation of Directrix

Equation of directrix or height of directrix above projection level is

$$y = \text{height of vertex} + VD$$

$$= \text{height of vertex} + \frac{1}{4}(L.R)$$

$$= \frac{V_0^2}{2g} + \frac{v_0^2}{2g} = \frac{v_0^2}{g}$$

$$\boxed{y = \frac{v_0^2}{g}}$$

Axis of Parabola of safety

$x = 0$ i.e y-axis is the axis of parabola of safety.

OR

We can find focus, vertex, Equation of directrix and equation of axis as under:
Equation of the parabola of safety is

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$$x^2 = -\frac{2V_0^2}{g}\left(y - \frac{V_0^2}{2g}\right)$$

$$(x-0)^2 = -\frac{2V_0^2}{g}\left(y - \frac{V_0^2}{2g}\right)$$

$$X^2 = -\frac{2V_0^2}{g}Y \quad \text{where } X = (x-0)^2$$

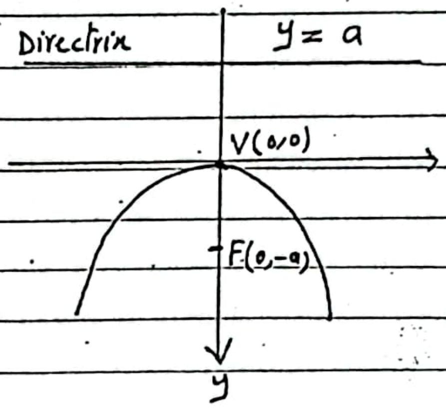
$$Y = y - \frac{V_0^2}{2g}$$

$\longrightarrow (2)$

Comparing this equation with standard equation of parabola

$$x^2 = -4ay$$

$\longrightarrow (3)$



we have

$$4a = \frac{2V_0^2}{g}$$

$$\Rightarrow a = \frac{V_0^2}{2g} \longrightarrow (4)$$

Vertex #

For vertex

$X = 0$ $\qquad$ $Y = 0$

$\Rightarrow x = 0$ $\qquad$ $y - \frac{V_0^2}{2g} = 0$

$$\Rightarrow y = \frac{V_0^2}{2g}$$

$\Rightarrow$ Vertex $= V\left(0, \frac{V_0^2}{2g}\right)$

Focus #

For focus

$X = 0$ $\qquad$ $Y = -a$

$\Rightarrow x = 0$ $\qquad$ $y - \frac{V_0^2}{2g} = -\frac{V_0^2}{2g}$ $\qquad$ putting $a = \frac{V_0^2}{2g}$ from (4)

$$\Rightarrow y = 0$$

Focus $= F(0,0)$ i.e at origin

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Equation of Directrix

Equation of directrix is

$$y = a$$

$$\Rightarrow y - \frac{V_0^2}{2g} = \frac{V_0^2}{2g} \qquad \text{putting } a = \frac{V_0^2}{2g} \text{ from (4)}$$

$$\Rightarrow y = \frac{V_0^2}{2g} + \frac{V_0^2}{2g}$$

$$y = \frac{V_0^2}{g}$$

Equation of axis

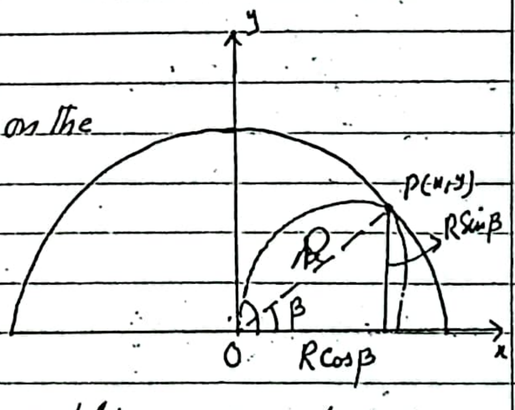
For equation of axis

$$X = 0$$

$$\Rightarrow x - 0 = 0$$

$x = 0$ i.e. y-axis is the equation of axis of parabola of safety.

Method-II for Derivation of Equation of Parabola of Safety



Let $P(x,y)$ be any point on the parabola of safety which surrounds all possible trajectories with same velocity of projection v_0. Since each point on the parabola of safety is a point at max distance from O and lies on some trajectory covered with velocity v_0 for some particular value of angle of projection α, therefore range on an imaginary inclined plane through P is OP and is max. Let $R = OP$ max range on an imaginary inclined plane through P of inclination β. Then

$$R = \frac{v_0^2}{g\cos^2\beta}\left[1 - \sin\beta\right]$$

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$$R = \frac{V_0^2}{g} \frac{[1 - \sin\beta]}{1 - \sin^2\beta}$$

$$R = \frac{V_0^2}{g(1+\sin\beta)}$$

$$R = \frac{\left(\frac{V_0^2}{g}\right)}{1+\sin\beta} \longrightarrow ①$$

Comparing it with the polar equation of parabola

$$r = \frac{l}{1+\sin\theta} \longrightarrow ②$$

where l is semi-latus-rectum and angle θ is measured from +ve x-axis and (r, θ) is polar point on this parabola and focus is at origin & it faces down.

We note that ① a parabola of safety in polar form with polar angle β, focus at origin and semi-latus-rectum equal to $\frac{V_0^2}{g}$.

We convert it into cartesian form.

from ①

$$R(1+\sin\beta) = \frac{V_0^2}{g}$$

$$R + R\sin\beta = \frac{V_0^2}{g}$$

But $y = R\sin\beta$

$x = R\cos\beta$, $R^2 = x^2+y^2$

$$\Rightarrow R + y = \frac{V_0^2}{g}$$

$$\Rightarrow R = \frac{V_0^2}{g} - y$$

squaring

$$R^2 = \left(\frac{V_0^2}{g} - y\right)^2$$

$$\Rightarrow x^2 + y^2 = \frac{V_0^4}{g^2} + y^2 - \frac{2V_0^2}{g}y$$

$$x^2 = -\frac{2V_0^2}{g}\left(y - \frac{V_0^2}{2g}\right)$$ which is required

equation of parabola of safety

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Summary of The Chapter

(1) Oblique # neither perpendicular nor horizontal.

(2) Projectile # A particle which is projected freely with an initial velocity in a direction oblique to the direction of gravity.

OR

A particle projected freely with an initial velocity making such an angle with the horizontal or the vertical that it covers horizontal as well as the vertical distance.

3) Equations for Projectile #

(a) # Horizontal and vertical velocities at any time t are

$$V_x = V_0 \cos\alpha$$
$$V_y = V_0 \sin\alpha - gt$$

(b) # Horizontal and vertical distances at any time t are

$$x = V_0 \cos\alpha \, t$$
$$y = V_0 \sin\alpha \, t - \frac{1}{2} g t^2$$

(c) # Equation of the path of the projectile is

$$y = x\tan\alpha - \frac{gx^2}{2V_0^2\cos^2\alpha}$$

(d) # path of the projectile is parabolic with equation

$$\left(x - \frac{V_0^2 \sin\alpha\cos\alpha}{g}\right)^2 = -\frac{2V_0^2\cos^2\alpha}{g}\left(y - \frac{V_0^2 \sin^2\alpha}{2g}\right)$$

$$\text{Vertex} = V\left(\frac{V_0^2 \sin 2\alpha}{2g}, \frac{V_0^2 \sin^2\alpha}{2g}\right)$$

$$L.R = \frac{2V_0^2\cos^2\alpha}{g}$$ (Reciprocal of the co-efficient of x^2 in the equation of path)

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equation of axis is

$$x = \frac{V_o^2 \sin\alpha \cos\alpha}{g} = \frac{V_o^2 \sin 2\alpha}{2g}$$

Focus of path $= F\left(\frac{V_o^2 \sin 2\alpha}{2g}, -\frac{V_o^2 \cos 2\alpha}{2g}\right)$

when $\alpha = 45°$, focus is

$$F\left(\frac{V_o^2}{2g}, 0\right)$$

Equation of Directrix is

$$y = \frac{V_o^2}{2g}$$

(e) # The Greatest Height

$$H = \frac{V_o^2 \sin^2\alpha}{2g} = \text{ordinate of vertex}$$

(f) # Time for Max Height #

time to reach maximum height is

$$t = \frac{V_o \sin\alpha}{g}$$

(g) # Total time of flight #

$$T = \frac{2 V_o \sin\alpha}{g}$$

(h) # Horizontal Range #

$$R = \frac{V_o^2 \sin 2\alpha}{g}$$

(i) # Maximum Horizontal Range $R_{max} = \frac{V_o^2}{g}$

4) # Direction of Projectile at any Time & at any Point #

(a) Direction of motion at any time t is found by

$\tan\theta = \frac{V_y}{V_x} = \frac{V_o \sin\alpha - gt}{V_o \cos\alpha}$, where θ is direction of velocity with horizontal

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(b) Direction of motion at any point is found by $\frac{dy}{dx}$ where $y = x\tan\alpha - \frac{gx^2}{2V_0^2\cos^2\alpha}$

5) # Magnitude of velocity at any height #

Velocity V of projectile at any height y is given by

$$V^2 = V_0^2 - 2gy$$

or

$$V^2 = 2g\left(\frac{V_0^2}{2g} - y\right)$$

$$= 2g\,(\text{depth of point below directrix})$$

$$= 2g\,(\text{Height of the directrix above pt})$$

6) Time for change in direction of velocity or Motion #

If velocity vector turns through an angle φ from its initial direction α, then time t for This change is given by

$$t = \frac{V_0 \sin\varphi}{g\cos(\alpha - \varphi)}$$

In particular time for direction to become ⊥ar to the origional direction is given by putting $\varphi = 90°$

$$t = \frac{V_0}{g\sin\alpha}$$

7) # K·E of Projectile at any height y

$$K{\cdot}E \text{ at any point } P(x,y) = \frac{1}{2}mV_0^2 - mgy$$

(To memorize Initial $K{\cdot}E = \frac{1}{2}mv_0^2$ and at height y (mgy) out of $\frac{1}{2}mv_0^2$ is changed into P.E so now $K{\cdot}E = \frac{1}{2}mv_0^2 - mgy$

8) # Minimum Initial Velocity to hit a Point (a, b) #

$$\text{Mim } V_0 = \sqrt{g\left(b + \sqrt{a^2+b^2}\right)}$$

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9) With Minimum velocity time to hit point (a, b)

$$t = \sqrt{\frac{2r}{g}} \quad \text{where } r = \sqrt{a^2+b^2}$$

10) Highest Point as origin #

Taking peak point of the path as origin, we have after time t

$$x = vt \qquad y = \frac{1}{2}gt^2$$

where v is velocity of particle at peak point.

Equation of path now is

$$y = \frac{gx^2}{2V^2}$$

11) # Projectile Launched from a Platform moving with velocity V

If the platform is moving towards target, there will be an increase in range of projectile given by

$$\frac{2VV_0 \sin\alpha}{g}$$

If the platform is moving away from the target, there will be a decrease in range given by

$$\frac{2VV_0 \sin\alpha}{g}$$

12) # Range on Inclined Plane #

(a) Range up inclined plane of inclination β is

$$R = \frac{2V_0^2 \sin(\alpha-\beta)\cos\alpha}{g\cos^2\beta}$$

α is angle of projection with horizontal

$$= \frac{V_0^2}{g\cos^2\beta}\left[\sin(2\alpha-\beta) - \sin\beta\right]$$

(b) Range down inclined plane is

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$$R = \frac{2V_0^2 \sin(\alpha+\beta)\cos\alpha}{g\cos^2\beta}$$

$$= \frac{V_0^2}{g\cos^2\beta}\left[\sin(2\alpha+\beta)+\sin\beta\right]$$

Maximum range up inclined plane is

$$R_{max} = \frac{V_0^2}{g\cos^2\beta}\left[1-\sin\beta\right] = \frac{V_0^2}{g(1+\sin\beta)}$$

Maximum range down inclined plane is

$$R_{max} = \frac{V_0^2}{g\cos^2\beta}\left[1+\sin\beta\right] = \frac{V_0^2}{g(1-\sin\beta)}$$

13)# Time of Flight on Inclined Plane

Time of flight up inclined plane is

$$T = \frac{2V_0\sin(\alpha-\beta)}{g\cos\beta}$$

Time of flight down inclined plane is

$$T = \frac{2V_0\sin(\alpha+\beta)}{g\cos\beta}$$

14)# Velocity at any time along & Perpendicular to inclined plane.

(a) For up inclined plane

Velocity parallel or along plane $V_{\parallel} = V_0\cos(\alpha-\beta) - g\sin\beta\, t$

Velocity ⊥ar to plane $V_{\perp} = V_0\sin(\alpha-\beta) - gt\cos\beta$

(b) Down inclined plane.

$$V_{\parallel} = V_0\cos(\alpha+\beta) + g\sin\beta\, t$$ change β into $-\beta$

$$V_{\perp} = V_0\sin(\alpha+\beta) - gt\cos\beta.$$

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15) Distances along and Perpendicular to the Inclined Plane

(a) up inclined plane

Distance along plane $= d_{\parallel} = V_0 \cos(\alpha-\beta)t - \frac{1}{2}gt^2\sin\beta$

Distance $\perp$ ar to plane $= d_{\perp} = V_0 \sin(\alpha-\beta)t - \frac{1}{2}gt^2\cos\beta$

(b) Down inclined plane

Distance along the plane $= V_0 \cos(\alpha+\beta) + \frac{1}{2}gt^2\sin\beta$

Distance $\perp$ ar to the plane $= V_0 \sin(\alpha+\beta) - \frac{1}{2}gt^2\cos\beta$

16) Direction of Motion on Striking Inclined plane

If θ is the angle which velocity makes with the horizontal on striking the horizontal, Then

$$\tan\theta = 2\tan\beta - \tan\alpha$$

17) # Condition for striking the inclined plane Perpendicularly

$$\cot\beta = 2\tan(\alpha-\beta)$$

18) # Parabola of Safety

A curve in vertical plane Through the point of projection bounding all trajecties of projectile with same vel of projection and touching each such trajectory at a point which is at max distance from the point of projection.

19) # Equation of Parabola of Safety

$$x^2 = -\frac{2V_0^2}{g}\left(y - \frac{V_0^2}{2g}\right)$$

Vertex $= \left(0, \frac{V_0^2}{2g}\right)$ $\quad$ L.R $= \frac{2V_0^2}{g}$

Focus $= F(0,0)$

Equation of directrix is $y = \frac{V_0^2}{g}$

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Example # A battleship is steaming ahead with speed V and a gun is mounted on the battleship so as to point straight backwords and is set at an elevation α. If V_0 is the speed of projection (relative to the gun), show that the range is

$$\frac{2V_0}{g}\sin\alpha\,(V_0\cos\alpha - V)$$. Prove also that the angle of elevation for maximum range is

$$\cos^{-1}\left(\frac{V+\sqrt{V^2+8V_0^2}}{4V_0}\right)$$

Sol# Horizontal vel of bullet relative to the gun $= v_0\cos\alpha$

Horizontal vel relative to ship $= v_0\cos\alpha - V$

Time of flight $= \frac{2V_0\sin\alpha}{g}$

R = range of bullet = (Time of flight) × (Horizontal vel)

$$= (v_0\cos\alpha - V)\times\frac{2V_0\sin\alpha}{g}$$

$$= \frac{v_0}{g}\left[v_0\sin 2\alpha - 2V\sin\alpha\right]$$

$$\frac{dR}{d\alpha} = \frac{2V_0}{g}\left[V_0\cos 2\alpha - V\cos\alpha\right] \longrightarrow (1)$$

$$\frac{d^2R}{d\alpha^2} = \frac{2V_0}{g}\left[-2V_0\sin 2\alpha + V\sin\alpha\right]$$

$$= -\frac{2V_0\sin\alpha}{g}\left(4V_0\cos\alpha - V\right) \longrightarrow (2)$$

For max range

$$\frac{dR}{d\alpha} = 0 \Rightarrow \frac{2V_0}{g}\left[-2V_0\cos 2\alpha + V\cos\alpha\right] = 0$$

$$\Rightarrow V_0\cos 2\alpha - V\cos\alpha = 0$$

$$\Rightarrow V_0(2\cos^2\alpha - 1) - V\cos\alpha = 0$$

$$\Rightarrow 2V_0\cos^2\alpha - V\cos\alpha - V_0 = 0$$

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$$\Rightarrow \cos\alpha = \frac{V \pm \sqrt{V^2 + 8V_0^2}}{4V_0}$$

$$\alpha = \cos^{-1}\left(\frac{V \pm \sqrt{V^2+8V_0^2}}{4V_0}\right)$$

At this value of α

$$\frac{d^2R}{d\alpha^2} = -\frac{2V_0 \sin\alpha}{g}\left[V \pm \sqrt{V^2+8V_0^2} - V\right]$$

Since $0 < \alpha < \pi \Rightarrow \sin\alpha$ is +ve

Hence $\frac{d^2R}{d\alpha^2}$ is negative when

$$\cos\alpha = \frac{V + \sqrt{V^2+8V_0^2}}{4V_0}$$

$$\alpha = \cos^{-1}\left[\frac{V + \sqrt{V^2+8V_0^2}}{4V_0}\right]$$

$\Rightarrow$ R is max at α given above

Example# An aeroplane is flying with uniform speed v_0 in an arc of a vertical circle of radius a, whose centre is at a height h vertically above a point O of ground. If a bomb is dropped from the aeroplane when at a height Y and strikes the grou. at O, show that Y satisfies the equation

$$KY^2 + Y(a^2 - 2hK) + K(h^2 - a^2) = 0$$

where $K = h + \frac{ga^2}{2V_0^2}$

Sol# Let aeroplane be at position O' when bomb is dropped. The bombe is launched in vertical plane with velocity v_0 fro O' and with angle of elevation $-\alpha$ (clockwise).

Let O' be taken as origin, then equation of the path of bomb is

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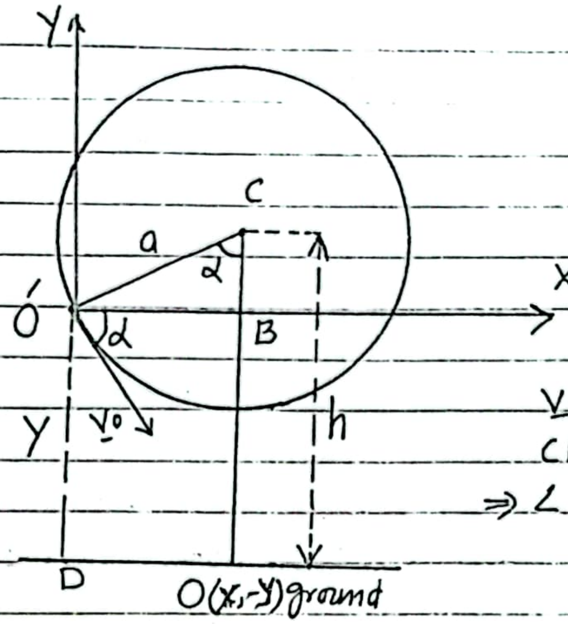

V_0 ⊥ar $O'C$
CB ⊥ar x-axis
$\Rightarrow \angle O'CB = \alpha$

$$y = x\tan(-\alpha) - \frac{gx^2}{2V_0^2}\sec^2\alpha$$

Since bomb hits $O(X,-Y)$, the target
Therefore

$$-Y = -X\tan\alpha - \frac{gX^2}{2V_0^2}\sec^2\alpha$$

$$Y = X\tan\alpha + \frac{gX^2}{2V_0^2}\sec^2\alpha \longrightarrow (1)$$

Now from $\Delta O'CB$, we have $\angle O'CB = \alpha$

$O'C = a$ (the radius)

$BC = h - Y$ where $h = OC$ i.e height of C from O

$$X = O'B = \sqrt{a^2 - (h-Y)^2}$$

$$\Rightarrow \tan\alpha = \frac{\sqrt{a^2-(h-Y)^2}}{h-Y}, \quad \sec\alpha = \frac{a}{h-Y}$$

using this values of X, tanα, secα in Eq (1)

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$$y = \sqrt{a^2-(h-y)^2}\cdot\frac{\sqrt{a^2-(h-y)^2}}{h-y} + \frac{g}{2V_0^2}\{a^2-(h-y)^2\}\cdot\frac{a^2}{(h-y)^2}$$

$$y = \frac{a^2-(h-y)^2}{h-y} + \frac{ga^2}{2V_0^2}\{a^2-(h-y)^2\}\cdot\frac{1}{(h-y)^2}$$

$$y(h-y)^2 = \{a^2-(h-y)^2\}(h-y) + \frac{ga^2}{2V_0^2}\{a^2-(h-y)^2\}$$

$$\Rightarrow \{a^2-(b-y)^2\}(h-y) + \frac{ga^2}{2V_0^2}\{a^2-(h-y)^2\} - y(h-y)^2 = 0$$

$$\Rightarrow \{a^2-(h-y)^2\}(h-y) + \frac{ga^2}{2V_0^2}\{a^2-h^2+2hy-y^2\} - y(h-y)^2 = 0$$

$$\{a^2-h^2-y^2+2hy\}(h-y) + \frac{ga^2}{2V_0^2}\{a^2-h^2+2hy-y^2\} - y(h-y)^2 = 0$$

$$(h-y)\{a^2-h^2-\cancel{y^2}+2hy-hy+\cancel{y^2}\} + \frac{ga^2}{2V_0^2}\{a^2-h^2+2hy-y^2\} = 0$$

$$(h-y)\{a^2-h^2+hy\} + \frac{ga^2}{2V_0^2}\{a^2-h^2\} + \frac{ga^2}{2V_0^2}(2hy-y^2) = 0$$

$$(a^2-h^2)h + h^2y - y\{a^2-h^2+hy\} + \frac{ga^2}{2V_0^2}\{a^2-h^2\} + \frac{ga^2}{2V_0^2}(2h-y)y = 0$$

$$\left(\frac{ga^2}{2V_0^2}+h\right)(a^2-h^2) + y\left\{\frac{ga^2}{2V_0^2}\cdot 2h - a^2 + 2h^2\right\} - y^2\left\{\frac{ga^2}{2V_0^2}+h\right\} = 0$$

$$\left(\frac{ga^2}{2V_0^2}+h\right)(a^2-h^2) + y\left[2h\left(\frac{ga^2}{2V_0^2}+h\right)-a^2\right] - y^2\left\{\frac{ga^2}{2V_0^2}+h\right\} = 0$$

putting $k = h + \frac{ga^2}{2V_0^2}$

Then $k(a^2-h) + y(2hk-a^2) - y^2k = 0$

which is the required Condition.

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Exercise

Q:1# A particle is projected at $t=0$ in a fixed vertical plane from a given point O with speed $\sqrt{2ga}$ of which vertical component is V. Show that $t = \frac{2a}{V}$ the particle is on fixed parabola (parabola of safety) that its path touches the parabola and that its direction of motion is then perpendicular to its initial direction.

Sol# Given that

Velocity of Projection $= v_0 = \sqrt{2ga}$

Let α be the angle of project and O be origin

V = Vertical Component of vel of projection

$= V_0 \sin\alpha$

$$V = \sqrt{2ga}\sin\alpha \longrightarrow (1)$$

$$\text{Time } t = \frac{2a}{V}$$

To Prove# (1) Particle touches parabola of safety at $t = \frac{2a}{V}$.

(2) Direction of motion of particle at time $t = \frac{2a}{V}$ is perpendicular to the direction of projection.

First we find $\sin\alpha$ and $\cos\alpha$ in terms of known quantities i.e we find α.

From (1)

$$\sin\alpha = \frac{V}{\sqrt{2ga}} \longrightarrow (2)$$

$$\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-\frac{V^2}{2ga}}$$

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$$\cos\alpha = \frac{\sqrt{2ga - V^2}}{\sqrt{2ga}} \longrightarrow (4)$$

Equation of Parabola of Safety when $V_0 = \sqrt{2ga}$

Equation of parabola of safety with given velocity of projection v_0 is

$$x^2 = -\frac{2v_0^2}{g}\left(y - \frac{v_0^2}{2g}\right)$$

putting $v_0 = \sqrt{2ga}$

$$x^2 = -\frac{2\cdot(2ga)}{g}\left(y - \frac{2ga}{2g}\right)$$

$$x^2 = -4a\,(y - a) \longrightarrow (5).$$

Let $P(x_1, y_1)$ be position of particle at $t = \frac{2a}{V}$. The particle will be on parabola of safety (5) if its position Co-ordinates (x_1, y_1) satisfy (5). First we find x_1 & y_1 at $t = \frac{2a}{V}$

At any time t, we have

$$x = V_0 \cos\alpha\, t$$

$$y = V_0 \sin\alpha\, t - \frac{1}{2} g t^2$$

at $t = \frac{2a}{V}$

$$x_1 = x\big|_{t=\frac{2a}{V}} = \sqrt{2ga}\cdot\frac{\sqrt{2ga - V^2}}{\sqrt{2ga}}\cdot\frac{2a}{V} - 1$$

$$= \frac{2a}{V}\sqrt{2ga - V^2}$$

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$$y_1 = y\big|_{t=\frac{2a}{V}} = \sqrt{2ga}\cdot\frac{V}{\sqrt{2ga}}\cdot\frac{2a}{V} - \frac{1}{2}g\left(\frac{2a}{V}\right)^2$$

$$= 2a - \frac{2ga^2}{V^2} = 2a\left(\frac{V^2 - ga}{V^2}\right)$$

Putting x_1 & y_1 in Equation (5)

$$L.H.S = x_1^2 = \left(\frac{2a}{V}\sqrt{2ga - V^2}\right)^2$$

$$= \frac{4a^2}{V^2}(2ga - V^2) \longrightarrow (6)$$

$$R.H.S = -4a(y_1 - a)$$

$$= -4a\left[2a\left(\frac{V^2 - ga}{V^2}\right) - a\right]$$

$$= -4a\left[\frac{2aV^2 - 2ga^2 - aV^2}{V^2}\right]$$

$$= \frac{-4a}{V^2}\left[aV^2 - 2ga^2\right]$$

$$= \frac{4a^2}{V^2}\left[2ga - V^2\right] \longrightarrow (7)$$

By (6) & (7)

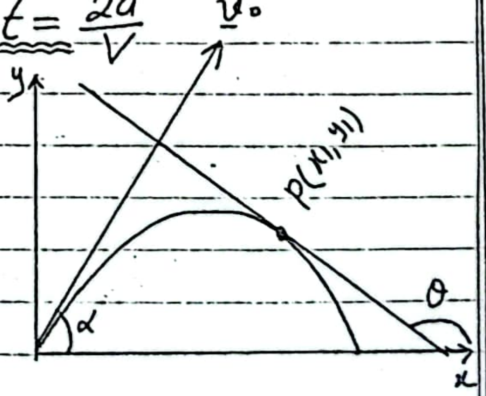
L.H.S = R.H.S

$\Rightarrow$ (x_1, y_1) lies on the parabola of safety

$\Rightarrow$ particle is on the parabola of safety at $t = \frac{2a}{V}$

Direction of Motion at $t = \frac{2a}{V}$

Since at $t = \frac{2a}{V}$ particle is on parabola of safety (5) at point $P(x_1, y_1)$, therefore direction of motion at $t = \frac{2a}{V}$ is along the tangent to (5) at $P(x_1, y_1)$



Differentiating (5) w.r.t x

$$2x = -4a\frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{x}{2a}$$

Let θ be be the inclination of tangent at P.
then

$$\tan\theta = \left.\frac{dy}{dx}\right|_{x_1=\frac{2a}{V}\sqrt{2ga-V^2}} = \frac{-\frac{2a}{V}\sqrt{2ga-V^2}}{2a}$$

$$= -\frac{1}{V}\sqrt{2ga-V^2}$$

$$= \text{Slope of direction at P} \longrightarrow (8)$$

Slope of the initial direction of motion is

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{\frac{V}{\sqrt{2ga}}}{\frac{\sqrt{2ga-V^2}}{\sqrt{2ga}}}$$

$$= \frac{V}{\sqrt{2ga-V^2}} \longrightarrow (9)$$

By (8) & (9)

$$\tan\theta\tan\alpha = \left(-\frac{1}{V}\sqrt{2ga-V^2}\right)\left(\frac{V}{\sqrt{2ga-V^2}}\right)$$

$$\tan\theta\tan\alpha = -1 \longrightarrow (10)$$

$\Rightarrow$ Two directions are perpendicular.
Hence direction of motion of particle at $t = \frac{2a}{V}$ is perpendicular to the initial direction of motion.

OR

from (10) $\tan\theta = -\frac{1}{\tan\alpha} = -\cot\alpha$

$$\tan\theta = \tan\left(\frac{\pi}{2}+\alpha\right)$$

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$\Rightarrow \theta = \frac{\pi}{2} + \alpha$

$\Rightarrow$ direction at p is $\perp$ ar to initial direction

Note If θ is angle of the direction of motion at $t = \frac{2a}{V}$, we can find $\tan\theta$ as under

$$\tan\theta = \frac{\text{Vertical vel at } t = \frac{2a}{V}}{\text{Horizontal vel at } t = \frac{2a}{V}}$$

$$= \frac{[V_y]_{t=\frac{2a}{V}}}{V_x|_{t=\frac{2a}{V}}}$$

$$= \frac{\{V_0 \sin\alpha - gt\}_{t=\frac{2a}{V}}}{V_0 \cos\alpha|_{t=\frac{2a}{V}}}$$

$$= \frac{\sqrt{2ga}\cdot\frac{V}{\sqrt{2ga}} - g\cdot\frac{2a}{V}}{\sqrt{2ga}\cdot\frac{\sqrt{2ga-V^2}}{\sqrt{2ga}}}$$

$$= \frac{V^2 - 2ga}{V}\cdot\frac{1}{\sqrt{2ga-V^2}}$$

$$= \frac{-(2ga-V^2)}{V\sqrt{2ga-V^2}} = -\frac{\sqrt{2ga-V^2}}{V}$$

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Q:2# Prove that the speed required to project a particle from a height h to fall a horizontal distance (a) from the point of projection is at least $\sqrt{g\left(\sqrt{a^2+h^2}-h\right)}$.

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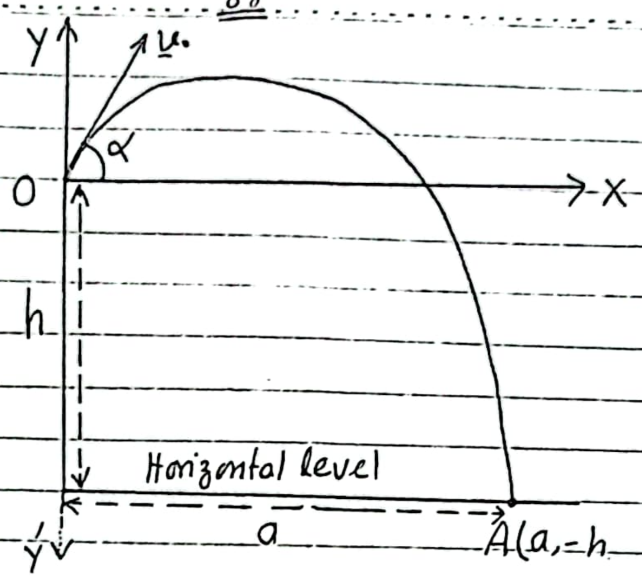
Sol #



Let the point of projection O at height h be origin and axes be as shown. Let α be angle of projection and v_0 be the velocity of projection.

Equation of the path is

$$y = x\tan\alpha - \frac{gx^2}{2V_0^2\cos^2\alpha}$$

$$= x\tan\alpha - \frac{gx^2}{2V_0^2}\sec^2\alpha$$

$$= x\tan\alpha - \frac{gx^2}{2V_0^2}(1+\tan^2\alpha)$$

$$gx^2\tan^2\alpha - 2V_0^2\tan\alpha\cdot x + gx^2 + 2V_0^2 y = 0 \rightarrow (1)$$

The Co-ordinates of target points are $(a, -h)$.

∵ Point $(a, -h)$ lies on (1)

$$\therefore ga^2\tan^2\alpha - 2V_0^2\tan\alpha\cdot a + ga^2 - 2V_0^2 h = 0$$

$$ga^2\tan^2\alpha - 2V_0^2 a\tan\alpha + ga^2 - 2V_0^2 h = 0 \rightarrow (2)$$

It is quadratic in $\tan\alpha$ and projectile will hit the point really if (2) has real roots (real directions) i.e if

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if Disc of ② ≥ 0

$$\Rightarrow 4V_0^4a^2 - 4ga^2(ga^2 - 2V_0^2h) \geq 0$$

$$\Rightarrow V_0^4 - g(ga^2 - 2V_0^2h) \geq 0$$

$$\Rightarrow V_0^4 - g^2a^2 + 2gV_0^2h \geq 0$$

$$\Rightarrow V_0^4 + 2gV_0^2h \geq g^2a^2$$

Adding g^2h^2 to both sides to complete square on L.H.S

$$V_0^4 + 2gV_0^2h + g^2h^2 \geq g^2a^2 + g^2h^2$$

$$\Rightarrow (v_0^2 + gh)^2 \geq g^2(a^2 + h^2)$$

$$\Rightarrow v_0^2 + gh \geq g\sqrt{a^2+h^2}$$

$$\Rightarrow v_0^2 \geq g\sqrt{a^2+h^2} - gh$$

$$\Rightarrow V_0^2 \geq g(\sqrt{a^2+h^2} - h)$$

$$\Rightarrow V_0 \geq \sqrt{g(\sqrt{a^2+h^2} - h)}$$

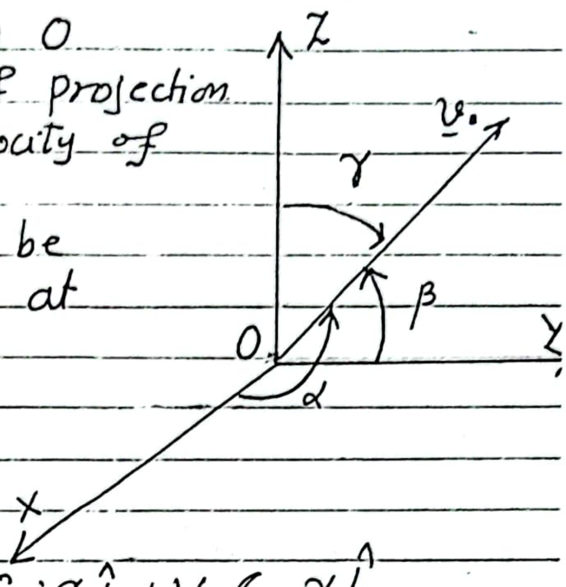
Hence the least required velocity to hit the target will be

$$V_0(\text{least}) = \sqrt{g(\sqrt{a^2+h^2} - h)}$$

Q:3 # Any number of particles are projected from same point at the same instant in various directions with speed v_0. Prove that at any subsequent time t, they will all be on a sphere of radius v_0t, and determine the motion of the centre of this sphere.

Sol # Let the origin O be the point of projection and $\underline{v}_0$ be the velocity of projection.

Let $P(x, y, z)$ be the position of partibles at instant t. Then



$$\underline{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\underline{v}_0 = V_0 \cos\alpha\,\hat{i} + V_0 \cos\beta\,\hat{j} + V_0 \cos\gamma\,\hat{k}$$

Since the particle are only under the influence of gravitational force, therefore

$$\underline{F} = -mg\hat{k}$$

$$m\ddot{r} = -mg\hat{k}$$

$$\ddot{r} = -g\hat{k}$$

Integrating w.r.t t

$$\underline{V} = \dot{\underline{r}} = \frac{dr}{dt} = -gt\hat{k} + \underline{a} \longrightarrow (1)$$

at $t = 0$ $\quad \underline{V} = \underline{V}_0$

$$\Rightarrow \underline{V}_0 = \underline{a}$$

putting this value of $\underline{a}$ in (1)

$$\dot{r} = -gt\hat{k} + \underline{V}_0$$

Again integrating w.r.t t

$$r = \underline{V}_0 t - \frac{1}{2}gt^2\hat{k} + \underline{b} \longrightarrow (2)$$

at $t = 0$ $\quad \underline{r} = 0$

$$\Rightarrow 0 = 0 + 0 + \underline{b}$$

putting this value of $\underline{b}$ in (2)

$$\underline{r} = \underline{V}_0 t - \frac{1}{2}gt^2\hat{k}$$

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$$\Rightarrow x\hat{i}+y\hat{j}+z\hat{k} = (V_o\cos\alpha\,\hat{i}+V_o\cos\beta\,\hat{j}+V_o\cos\gamma\,\hat{k})t - \frac{1}{2}gt^2\hat{k}$$

Comparing Co-efficients of $\hat{i},\hat{j},\hat{k}$

$$x = V_o\cos\alpha . t \longrightarrow (3)$$

$$y = V_o\cos\beta . t \longrightarrow (4)$$

$$z = V_o\cos\gamma . t - \frac{1}{2}gt^2$$

$$\Rightarrow z+\frac{1}{2}gt^2 = V_o\cos\gamma . t \longrightarrow (5)$$

squaring and adding (3), (4), (5)

$$x^2+y^2+(z+\frac{1}{2}gt^2)^2 = V_o^2t^2(\cos^2\alpha+\cos^2\beta+\cos^2\gamma)$$

$$(x-0)^2+(y-0)^2+(z-(-\frac{1}{2}gt^2))^2 = (V_ot)^2 \longrightarrow (6)$$

$\because \cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$

It represents a sphere with centre at $(0,0,-\frac{1}{2}gt^2)$ and radius V_ot

As t varies $-\frac{1}{2}gt^2$ becomes more and more $-ve$. Therefore centre moves vertically downwards along z-axis.

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Q:4# A shell bursts on contact with the ground and pieces from it fly in all directions with all speeds upto 80 feet per second. Prove that a man 100 feet away is in danger for $\frac{5}{\sqrt{2}}$ seconds.

Sol# Since range of projectile depends upon the velocity of projectile, therefore greater velocity means greater range and greater time of flight

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Hence we consider the possibility to hit the man for piece having max velocity $v_0 = 80$ ft/sec

Since the man is 100 ft away from the shell, we take

$$R = 100$$

Hence $v_0 = 80$ $\quad R = 100 \quad g = 32$ ft/sec^2

$$R = \frac{v_0^2 \sin 2\alpha}{g}$$

$$\Rightarrow 100 = \frac{(80)^2}{32} \times \sin 2\alpha$$

$$\sin 2\alpha = \frac{100 \times 32}{80 \times 80} = \frac{1}{2}$$

$$\Rightarrow 2\alpha = 30^\circ \text{ or } 180^\circ - 30^\circ = 150^\circ$$

$$\Rightarrow \alpha = 15^\circ \text{ or } 75^\circ$$

These are two directions for the piece to hit the man. Let t_1, t_2 the times of flight for these directions. Then

$$t_1 = \frac{2v_0 \sin\alpha}{g} = \frac{2(80)\sin 15^\circ}{32} = 5\sin 15^\circ$$

$$t_2 = \frac{2v_0 \sin\alpha}{g} = \frac{2(80)\sin 75^\circ}{32} = 5\sin 75^\circ$$

Hence time for danger

$$= t_2 - t_1 = 5.(\sin 75^\circ - \sin 15^\circ)$$

$$= 5 \cdot 2\cos\left(\frac{75^\circ + 15^\circ}{2}\right)\sin\left(\frac{75^\circ - 15^\circ}{2}\right)$$

$$= 10 \cos 45^\circ \sin 30^\circ$$

$$= 10 \times \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{5}{\sqrt{2}} \text{ sec}$$

OR

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Since Max range depends upon velocity of projection, therefore we find max range of a piece with max velocity $v_0 = 80\,ft/sec$ and time for that range.

$$R_{max} = \frac{v_0^2}{g} = \frac{(80)^2}{32} = 200\,ft$$

⇒ A man 100 ft away is within the range such a piece.

Now time of flight for this max range

$$T = \frac{2V_0 \sin\alpha}{g}$$

$$= \frac{2 \times 80 \times \sin 45^\circ}{32} \quad \because \text{for max range } \alpha = 45^\circ$$

$$= 5 \times \frac{1}{\sqrt{2}} = \frac{5}{\sqrt{2}} \text{ seconds.}$$

Thus a man 100 ft away will be in danger for $\frac{5}{\sqrt{2}}$ seconds.

Q:5*# An aeroplane flying with a constant speed v_0 and at a constant height h. Show that, if a gun is fired point blank (adj (of a shot) fired with a gun very close to the target) at the aeroplane after it has passed directly over the gun when its angle of elevation as seen from the gun is α, the shell will hit the aeroplane provided that

$$2(V\cos\alpha - v_0)\,v_0 \tan^2\alpha = gh$$

where V is the initial speed of the shot, the path being assumed parabolic.

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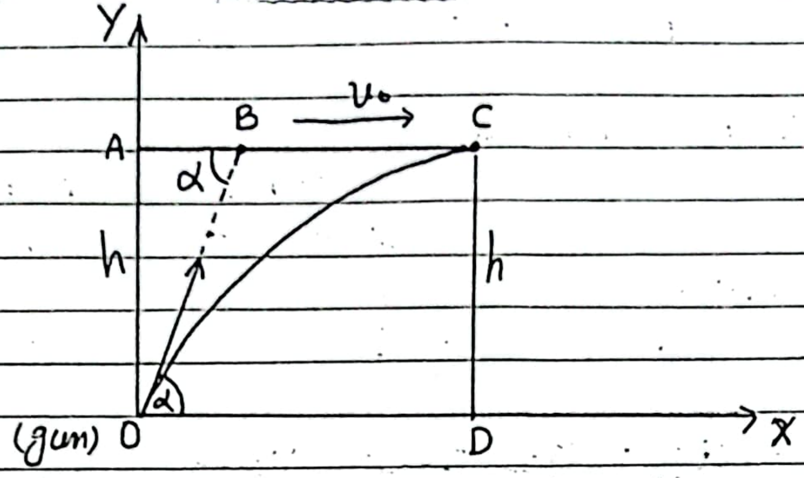
Sol # Method # 1



Let the position gun be O which is considered as origin. Suppose the gun is fired when the aeroplane is at point B (a point). Since the path of the shell is parabolic, therefore suppose it hits the plane at point C after time t.

Then from ΔOAB

$$\frac{h}{AB} = \tan\alpha$$

$$\Rightarrow AB = h\cot\alpha \longrightarrow (1)$$

Motion of Shell

Horizontal distance covered by shell during time t is

$$AC = V\cos\alpha \cdot t$$

$$AB + BC = V\cos\alpha \cdot t$$

putting $AB = h\cot\alpha$ from (1)

$$h\cot\alpha + BC = V\cos\alpha \cdot t \longrightarrow (2)$$

Vertical distance covered by shell is h and so

$$h = V\sin\alpha\, t - \frac{1}{2}gt^2 \longrightarrow (3)$$

Motion of Plane

Since the plane is moving with constant velocity v_0 and BC is distance covered

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by it, therefore

$$BC = v_o t \longrightarrow (4)$$

Required Condition

To find the required condition we are to eliminate unknowns BC, t from equations (2), (3) & (4)

putting (4) in (2)

$$h\cot\alpha + v_o t = V\cos\alpha . t$$

$$\Rightarrow h\cot\alpha = (V\cos\alpha - V_o)t$$

$$\Rightarrow t = \frac{h\cot\alpha}{V\cos\alpha - V_o}$$

putting this value of t in (3)

$$h = V\sin\alpha \left(\frac{h\cot\alpha}{V\cos\alpha - V_o}\right) - \frac{1}{2}g\left(\frac{h\cot\alpha}{V\cos\alpha - V_o}\right)^2$$

$$h = \frac{Vh\cos\alpha}{V\cos\alpha - V_o} - \frac{gh^2\cot^2\alpha}{2(V\cos\alpha - V_o)^2}$$

Multiplying by $2(V\cos\alpha - V_o)^2$

$$2h(V\cos\alpha - V_o)^2 = 2Vh(V\cos\alpha - V_o).\cos\alpha - gh^2\cot^2\alpha$$

$$2(V\cos\alpha - V_o)^2 = 2V(V\cos\alpha - V_o).\cos\alpha - gh\cot^2\alpha$$

$$\Rightarrow gh\cot^2\alpha = 2V(V\cos\alpha - V_o)\cos\alpha - 2(V\cos\alpha - V_o)^2$$

$$= 2[V^2\cos^2\alpha - VV_o\cos\alpha - V^2\cos^2\alpha - V_o^2 + 2VV_o\cos\alpha]$$

$$= 2(VV_o\cos\alpha - V_o^2) = 2V_o(V\cos\alpha - V_o)$$

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$$gh = 2(V\cos\alpha - V_0)V_0\tan^2\alpha$$

which is required condition

Q:6*# Find the range of a rifle bullet when α is the elevation of projection and v_0 the speed. Show that, if the rifle is fired with the same elevation and speed from a car travelling with speed V toward the target the range will be increased by $\frac{2V_0 V\sin\alpha}{g}$

Sol# Initial Horizontal Vel $= v_x = V_0\cos\alpha$

Initial Vertical Vel $= v_y = V_0\sin\alpha$

If (x, y) is position of the bullet (when) after time t, then

$$x = V_0\cos\alpha \cdot t \longrightarrow (1)$$

$$y = V_0\sin\alpha\, t - \frac{1}{2}gt^2 \longrightarrow (2)$$

For Horizontal range time of flight is time taken for projection level to projection level i.e time when

$$y = 0$$

$$\Rightarrow V_0\sin\alpha \cdot t - \frac{1}{2}gt^2 = 0$$

$$\Rightarrow t\left(V_0\sin\alpha - \frac{1}{2}gt\right) = 0$$

$$\Rightarrow t = 0 \quad \text{or} \quad V_0\sin\alpha - \frac{1}{2}gt = 0$$

$$\text{or} \quad t = \frac{2V_0\sin\alpha}{g}$$

$t = 0$ corresponds to the initial position of the projection

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Time of flight $= \frac{2V_0 \sin\alpha}{g}$

Horizontal range = R = Distance covered along horizontal in time $\frac{2V_0 \sin\alpha}{g}$

= (Horizontal vel)(Time of flight)

$R = (V_0 \cos\alpha)\left(\frac{2V_0 \sin\alpha}{g}\right)$

$= \frac{2V_0^2 \cos\alpha \sin\alpha}{g}$

In case the rifle is fired from a car moving with speed V towards the target, then the horizontal velocity increases by V. Hence

Horizontal Velocity $= V_0 \cos\alpha + V$

Horizontal Range $= R' =$ (Horizontal vel)(time of flight)

$= (V_0 \cos\alpha + V)\left(\frac{2V_0 \sin\alpha}{g}\right)$

$= \frac{2V_0^2 \sin\alpha \cos\alpha}{g} + \frac{2VV_0 \sin\alpha}{g}$

$= R + \frac{2VV_0 \sin\alpha}{g}$

So increase in range $= \frac{2VV_0 \sin\alpha}{g}$

Q:7**# The range of a rifle bullet is 1200 yards when α is the elevation of projection. Show that if the rifle is fired with the same elevation from a car travelling at 10 miles/hour towards the target, the range will be increased

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by $220\sqrt{\tan\alpha}$ feet.

Sol# Given

Range $= R = 1200$ yards
$= 1200 \times 3$ feet
$= 3600$ feet

velocity of car $= V = 10$ miles/hour

$= \dfrac{10 \times 5280}{3600}$ ft/sec

$= \dfrac{44}{3}$ ft/sec

one mile $= 1760$ yard
1 yard $= 3$ feet
⇒
one mile $= 1760 \times 3$ fts $= 5280$ fts

Let v_0 be the velocity of projection of bullet. Then

$$R = \frac{V_0^2 \sin 2\alpha}{g}$$

$$3600 = \frac{V_0^2 \sin 2\alpha}{g}$$

$$v_0^2 = \frac{3600\, g}{\sin 2\alpha}$$

$$v_0 = \sqrt{\frac{3600\, g}{\sin 2\alpha}}$$

$$\text{increase in range} = \frac{2 V v_0 \sin\alpha}{g}$$

$$= 2 \cdot \sqrt{\frac{3600\, g}{\sin 2\alpha}} \times \frac{44}{3} \times \frac{\sin\alpha}{g}$$

$$= \frac{2}{32} \cdot \sqrt{\frac{3600 \times 32 \times \sin^2\alpha}{2 \sin\alpha \cos\alpha}} \times \frac{44}{3}$$

$$= \frac{2}{32_{16}} \times \frac{44}{3} \times 240^{80^5} \sqrt{\tan\alpha}$$

$$= 220\sqrt{\tan\alpha} \quad \text{As Required.}$$

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Q:8* # Determine the maximum possible range for a projectile fired from a Canon having muzzle velocity v_0 and Prove that the height reached in this case is $\frac{v_0^2}{4g}$

Sol # Let α be the angle of projection.

Horizontal vel = $V_0 \cos\alpha$

Vertical vel = $V_0 \sin\alpha$

Vertical distance at any time t is

$$y = V_0 \sin\alpha \, t - \frac{1}{2}gt^2$$

for time of flight putting $y = 0$

$$\Rightarrow \quad V_0 \sin\alpha \, t - \frac{1}{2}gt^2 = 0$$

$$\Rightarrow \quad t(V_0 \sin\alpha - \frac{1}{2}gt) = 0$$

$$\Rightarrow \quad t = 0 \quad \text{or} \quad V_0 \sin\alpha - \frac{1}{2}gt = 0$$

$t = 0$ corresponds to initial projection. Hence

$$V_0 \sin\alpha - \frac{1}{2}gt = 0$$

$$t = \frac{2V_0 \sin\alpha}{g}$$

Horizontal Range = (Horizontal vel)(Time of flight)

$$R = \frac{V_0 \cos\alpha \cdot 2V_0 \sin\alpha}{g}$$

$$R = \frac{V_0^2 \sin 2\alpha}{g}$$

R is max. when $\sin 2\alpha = 1 \Rightarrow 2\alpha = 90^\circ$

$\Rightarrow \alpha = 45^\circ$

$$R_{max} = \frac{V_0^2}{g}$$

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Height Reached

Consider vertical Motion
Initial vertical vel = $V_0 \sin\alpha$
acc = $-g$
final vertical vel for Height reach = 0
distance = Height reached $y = H = ?$

$$2(-g)H = V^2 - u^2$$
$$= 0 - V_0^2 \sin^2\alpha$$
$$H = \frac{V_0^2 \sin^2\alpha}{2g}$$

But when projectile covers max horizontal range then $\alpha = 45^\circ$ and Height reached in this case is

$$H = \frac{V_0^2 \sin^2 45^\circ}{2g} = \frac{V_0^2\left(\frac{1}{\sqrt{2}}\right)^2}{2g}$$

$$H = \frac{V_0^2}{4g}$$

Q:10 # A canon has its maximum range R. Prove that (a) the height reached $\frac{1}{4}R$, and (b) the time of flight is $\sqrt{\frac{2R}{g}}$

Sol # Let α be the angle of elevation and v_0 be the initial velocity of projection.
Given Max Range = R.

$$R_{max} = \frac{v_0^2}{g}$$

$$R = \frac{v_0^2}{g} \Rightarrow v_0^2 = Rg \longrightarrow ①$$

$\because$ Range is max $\therefore \alpha = 45^\circ$

Height Reached = $H = \frac{v_0^2}{4g} = \frac{Rg}{4g} = \frac{1}{4}R$

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Time of flight $= \dfrac{2V_0 \sin\alpha}{g}$

$= \dfrac{2V_0 \sin 45^\circ}{g}$

$= \dfrac{2\sqrt{gR}\cdot\frac{1}{\sqrt{2}}}{g} = \dfrac{\sqrt{2}\sqrt{R}}{\sqrt{g}}$

$= \sqrt{\dfrac{2R}{g}}$

Q:11# A projectile having horizontal range R, reaches max. Height H. Prove that it must have beeen launched with (a) an initial speed equal to $\sqrt{\dfrac{g(R^2+16H^2)}{8H}}$ and (b) at an angle with the horizontal given by $\sin^{-1}\left(\dfrac{4H}{\sqrt{R^2+16H^2}}\right)$

Sol# Given that

Max. Height $= H$

Horizontal range $= R$

To find# Velocity of Projection = ?

Angle of elevation = ?

Let V_0 be the velocity of projection and α be the angle of elevation.

Horizontal range $= R = \dfrac{2V_0^2 \sin\alpha\cos\alpha}{g}$ $\longrightarrow$ (1)

Max height $= H = \dfrac{V_0^2 \sin^2\alpha}{2g}$ $\longrightarrow$ (2)

Dividing (2) by (1)

$$\frac{H}{R} = \frac{V_0^2 \sin^2\alpha}{2g} \times \frac{g}{2V_0^2 \sin\alpha\cdot\cos\alpha} = \frac{\tan\alpha}{4}$$

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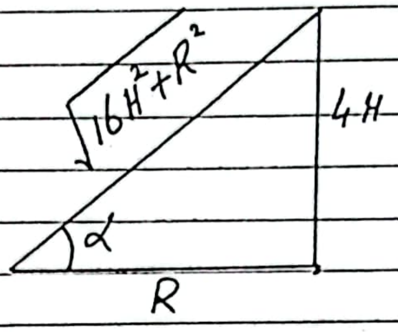
$$\Rightarrow \tan\alpha = \frac{4H}{R}$$

$$\sin\alpha = \frac{4H}{\sqrt{16H^2 + R^2}}$$

$$\alpha = \sin^{-1}\left(\frac{4H}{\sqrt{16H^2 + R^2}}\right)$$



Also from ②

$$v_0^2 = \frac{2gH}{\sin^2\alpha} = 2gH \cdot \frac{R^2 + 16H^2}{16H^2}$$

$$= \frac{g(R^2 + 16H^2)}{8H}$$

$$v_0 = \sqrt{\frac{g(R^2 + 16H^2)}{8H}}$$

OR

Squaring ①

$$R^2 = \frac{4V_0^2 \cdot V_0^2 \sin^2\alpha \cdot \cos^2\alpha}{g^2}$$

$$= \left(\frac{4V_0^2\cos^2\alpha}{g}\right)\left(\frac{V_0^2\sin^2\alpha}{g}\right) \longrightarrow ③$$

from ② $\frac{V_0^2\sin^2\alpha}{g} = 2H$

$$\Rightarrow 2H = \frac{V_0^2\sin^2\alpha}{g} \longrightarrow ④$$

using in ③

$$R^2 = \frac{4V_0^2\cos^2\alpha}{g} \cdot 2H$$

$$R^2 = \frac{8V_0^2\cos^2\alpha \cdot H}{g}$$

$$\Rightarrow \frac{R^2}{8H} = \frac{V_0^2\cos^2\alpha}{g} \longrightarrow ⑤$$

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Adding ④ & ⑤

$$2H + \frac{R^2}{8H} = \frac{V_o^2}{g}$$

$$\frac{16H^2 + R^2}{8H} = \frac{V_o^2}{g}$$

$$V_o^2 = g\left(\frac{16H^2 + R^2}{8H}\right)$$

$$V_o = \sqrt{g\left(\frac{16H^2 + R^2}{8H}\right)}$$

Dividing ④ by ⑤

$$\tan^2\alpha = \frac{16H^2}{R^2} = \left(\frac{4H}{R}\right)^2$$

$$\Rightarrow \tan\alpha = \frac{4H}{R}$$

$$\sin\alpha = \frac{4H}{\sqrt{R^2 + 16H^2}}$$

$$\alpha = \sin^{-1}\left(\frac{4H}{\sqrt{R^2 + 16H^2}}\right)$$

Q:12 # A projectile is launched at an angle α from a cliff of height H above sea level. If it falls into the sea at a distance D from the base of the cliff, prove that the max height above sea level is

$$H + \frac{D^2\tan^2\alpha}{4(H + D\tan\alpha)}$$

Sol # Let point O on the cliff of Height H above sea level be the point of projection and is taken as origin.

Let v_o be the velocity of projection.

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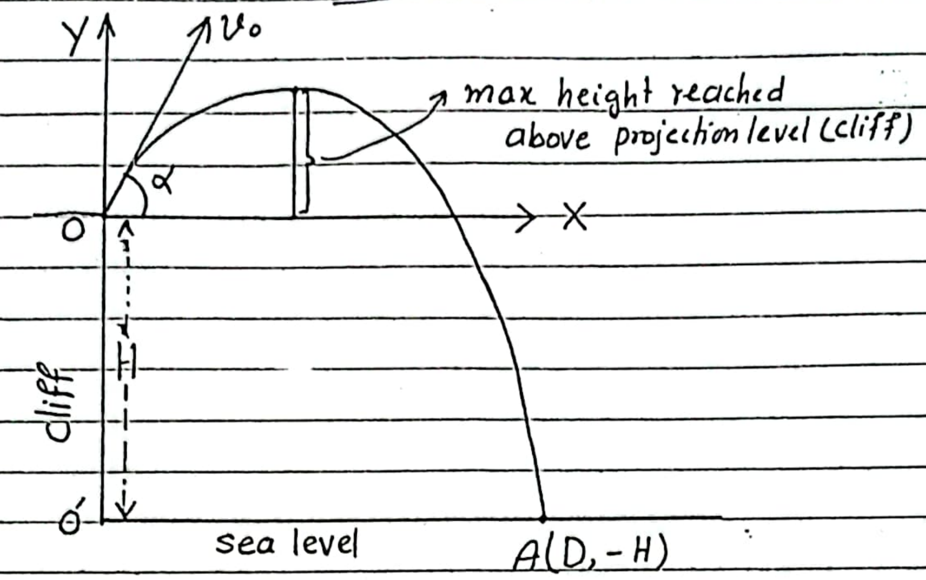

Maximum height reached above sea level

= (Height of cliff) + (Max height above projection level)

$$= H + \frac{V_o^2 \sin^2\alpha}{2g} \longrightarrow (1)$$

Here all quantities are known except v_o. So we find v_o and put in it.

Equation of the trajectory is

$$y = x\tan\alpha - \frac{gx^2}{2V_o^2\cos^2\alpha} \longrightarrow (2)$$

The co-ordinates of the point where the projectile falls into sea are A(D, −H).

Since, the point A lies on (2), therefore

$$-H = D\tan\alpha - \frac{gD^2}{2V_o^2}\sec^2\alpha$$

$$\Rightarrow \frac{gD^2}{2V_o^2}\sec^2\alpha = H + D\tan\alpha$$

$$\Rightarrow v_o^2 = \frac{gD^2\sec^2\alpha}{2(H + D\tan\alpha)} = \frac{gD^2}{2\cos^2\alpha(H + D\tan\alpha)}$$

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putting value of v_0 in ①

Max Height above sea level

$$= H + \frac{gD^2}{2\cos^2\alpha(H + D\tan\alpha)} \times \frac{\sin^2\alpha}{2g}$$

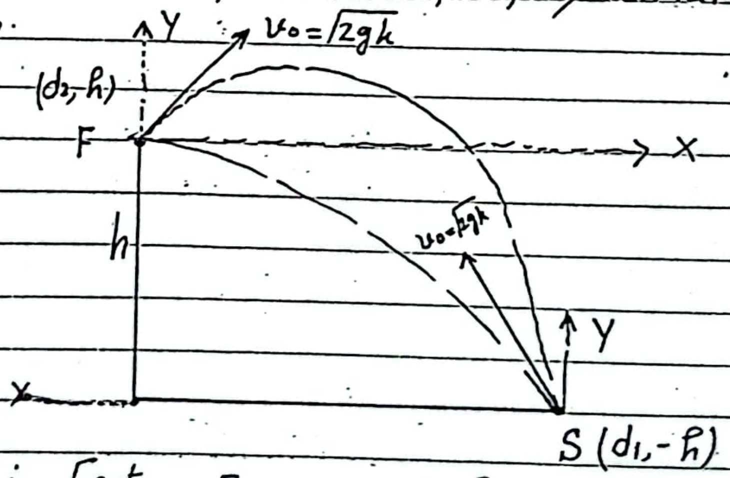
$$= H + \frac{D^2\tan^2\alpha}{4(H + D\tan\alpha)}$$

Q:13#** A fort and ship are both armed with guns which give their projectiles a muzzle velocity $\sqrt{2gk}$, and the guns in the fort are at a height h above the guns in the ship. If d_1 & d_2 are the greatest horizontal ranges at which the fort and ship, respectively, can engage, prove that

$$\frac{d_1}{d_2} = \sqrt{\frac{k+h}{k-h}}$$

Sol# **Note#** We know that for a given vel of projection a point which is at max. distance from point of projection and is on the trajectory of projectile will lie on parabola of safety with that vel of projection.

Let F be fort and S be ship.



For guns in Fort

For guns in fort if we consider F as origin and point of projection

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, the Point $(d_1, -h)$ in ship will be on the trajectory and will at max distance from point of projection $F(0,0)$ because d_1 is the greatest horizontal range at which fort can engage. Hence point $(d_1, -h)$ lies on the parabola of safety in vertical plane through F. for vel $v_0 = \sqrt{2gk}$

Equation of parabola of safety is

$$x^2 = -\frac{2v_0^2}{g}\left(y - \frac{v_0^2}{2g}\right)$$

putting $v_0 = \sqrt{2gk}$

parabola of safety is

$$x^2 = -\frac{2(\sqrt{2gk})^2}{g}\left(y - \frac{(\sqrt{2gk})^2}{2g}\right)$$

$$x^2 = -4k(y-k) \longrightarrow (1)$$

$\because$ $(d_1, -h)$ lies on it

$$\therefore \quad d_1^2 = -4k(-h-k)$$

$$\Rightarrow \quad d_1^2 = 4k(h+k) \longrightarrow (2)$$

For Guns in Ship # For guns in ship considering S as origin and point of projection, the point (d_2, h) in fort will be on trajectory with vel $v_0 = \sqrt{2gk}$ and will be at max distance from point of projection because d_2 is the greatest horizontal range at which ship can engage. Therefore (d_2, h) will lie on parabola of safety with $v_0 = \sqrt{2gk}$ which is

$$x^2 = -4k(y-k)$$

$\because$ (d_2, h) lies on it

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$$\therefore \quad d_2^2 = -4k(h-k)$$

$$d_2^2 = 4k(k-h) \longrightarrow (3)$$

dividing (2) by (3)

$$\frac{d_1^2}{d_2^2} = \frac{k+h}{k-h}$$

$$\frac{d_1}{d_2} = \sqrt{\frac{k+h}{k-h}}$$

Method-II

Concept to be used

With a given velocity of projection v_0, in general there are to direction to hit a point (a, b) and in this case

$$v_0^2 > g(b+\sqrt{a^2+b^2})$$

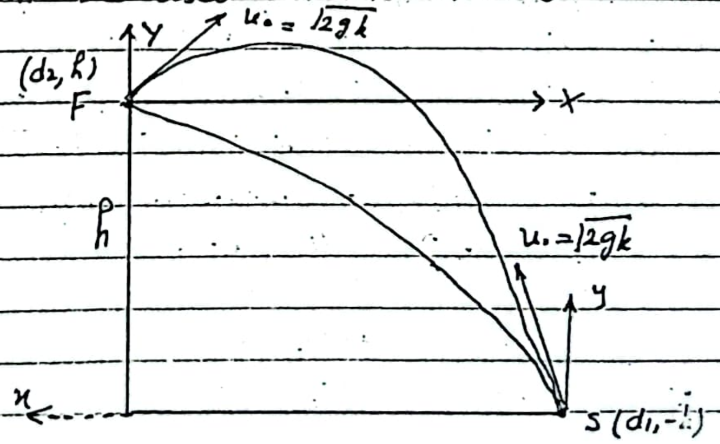
But when the point (a, b) is such that a, is equal to maximum horizontal range, then there is only one direction to hit (a, b) and

$$v_0^2 = g(b+\sqrt{a^2+b^2})$$

This idea will be used in this method.

Sol #

Let F be fort at height h above the ship S.



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For Guns in Fort

velocity of projection $= v_0 = \sqrt{2gk}$

Since guns in fort can engage at the greatest horizontal range d_1, therefore taking F as point of projection and origin, there will be one direction to hit point $(d_1, -h)$ and we have

$$v_0 = \sqrt{g(\sqrt{d_1^2+h^2} - h)}$$

$$\sqrt{2gk} = \sqrt{g(\sqrt{d_1^2+h^2} - h)}$$

Squaring

$$2gk = g(\sqrt{d_1^2+h^2} - h)$$

$$\Rightarrow 2k = \sqrt{d_1^2+h^2} - h$$

$$\Rightarrow 2k+h = \sqrt{d_1^2+h^2}$$

Squaring

$$\Rightarrow (2k+h)^2 = d_1^2+h^2$$

$$4k^2+h^2+4hk = d_1^2+h^2$$

$$d_1^2 = 4k(k+h)$$

$$d_1 = 2\sqrt{k(k+h)} \longrightarrow (1)$$

For Guns in Ship

Taking ship as origin

we have

$$v_0^2 = g(\sqrt{d_2^2+h^2} + h$$

putting $v_0 = \sqrt{2gk}$

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$$2gk = g(\sqrt{d_2^2+h^2}+h)$$

$$\Rightarrow \quad 2k = \sqrt{d_2^2+h^2}+h$$

$$\Rightarrow \quad 2k-h = \sqrt{d_2^2+h^2}$$

squaring

$$(2k-h)^2 = d_2^2+h^2$$

$$4k^2+h^2-4kh = d_2^2+h^2$$

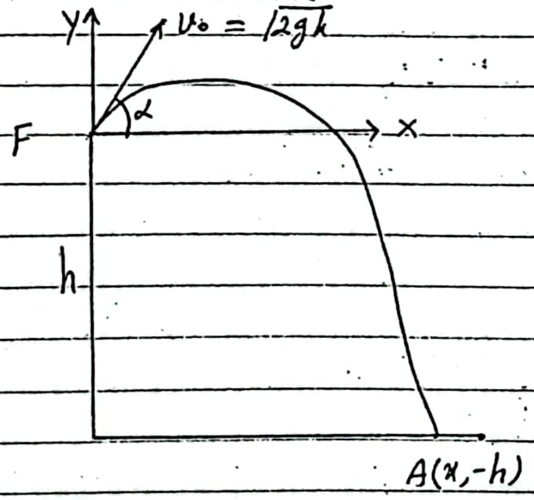
$$d_2^2 = 4k(k-h)$$

$$d_2 = 2\sqrt{k(k-h)} \longrightarrow (2)$$

dividing (1) by (2)

$$\frac{d_1}{d_2} = \sqrt{\frac{k+h}{k-h}}$$

Method-III



Let F(0,0) be the position of the point of projection of the guns in the fort. Since the guns in the fort are at height h. If a a shell from F hits

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at $A(x, -h)$, then A lies on the path

$$y = x\tan\alpha - \frac{gx^2\sec^2\alpha}{2V_0^2}$$

So

$$-h = x\tan\alpha - \frac{gx^2\sec^2\alpha}{2V_0^2} \longrightarrow (1)$$

We want that value of α which makes x maximum. So $\frac{dx}{d\alpha} = 0$

Differentiating (1) w.r.t α, we have

$$0 = x\sec^2\alpha + \tan\alpha\frac{dx}{d\alpha} - \frac{g\sec^2\alpha}{2V_0^2}\cdot 2x\cdot\frac{dx}{d\alpha} - \frac{gx^2\cdot 2\sec\alpha\cdot\sec\alpha\tan\alpha}{2V_0^2}$$

putting $\frac{dx}{d\alpha} = 0$, we have

$$x\sec^2\alpha\left(1 - \frac{gx\tan\alpha}{v_0^2}\right) = 0$$

$$\Rightarrow \tan\alpha = \frac{v_0^2}{gx}$$

$\because x\sec^2\alpha = 0$
$\Rightarrow x = 0$ or $\sec\alpha = 0$
not possible or $\frac{1}{\cos\alpha} = 0$
or $\cos\alpha = \infty$
which is not possible

$$\Rightarrow \tan\alpha = \frac{2gk}{gx}$$

$$= \frac{2k}{x}$$

putting value of $\tan\alpha$ in (1)

$$-h = x\cdot\frac{2k}{x} - \frac{gx^2}{2\cdot 2gk}(1+\tan^2\alpha)$$

$$\text{or } -h = 2k - \frac{x^2}{4k}\left(1 + \frac{4k^2}{x^2}\right)$$

$$= 2k - \frac{x^2}{4k} - k$$

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or $h+k = \frac{x^2}{4k}$

$\Rightarrow x = \sqrt{4k(h+k)}$

$\Rightarrow x = d_1 = \sqrt{4k(h+k)} \longrightarrow (2)$

Replacing h by $-h$ we get the maximum range for the guns in the ship i.e

$d_2 = \sqrt{4k(k-h)} \longrightarrow (3)$

dividing (2) by (3)

$$\frac{d_1}{d_2} = \sqrt{\frac{k+h}{k-h}}$$

Note We can also find the angle of projection from the fort for range is

$$\tan^{-1}\left[\sqrt{\frac{k}{h+k}}\right] \text{ as}$$

For maximum range of the guns on the fort

$$x = d_1 = \sqrt{4k(k+h)} = 2\sqrt{k(k+h)}$$

and $\tan\alpha = \frac{2k}{d_1} = \frac{2k}{2\sqrt{k}\sqrt{k+h}}$

$$= \frac{\sqrt{k}}{\sqrt{k+h}}$$

$$\alpha = \tan^{-1}\left[\sqrt{\frac{k}{h+k}}\right]$$

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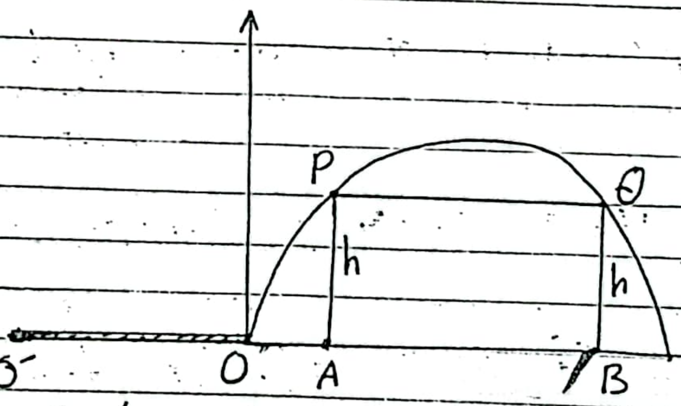
Q:14 # From a gun placed on a horizontal plane, which fire a shell with speed $\sqrt{2gH}$, it is required to (through) throw a shell over a wall of height h, and the elevation of gun cannot exceed $\alpha < 45°$. Show that this will be possible only when $h < H\sin^2\alpha$ and that if this condition be satisfied, the gun must be fired from with a strip of the plane whose breadth is

$$4\cos\alpha\sqrt{H(H\sin^2\alpha - h)}$$

Sol # Velocity of projection $= v_0 = \sqrt{2gH}$

Since the parabolic path is symmetric about the axis, therefore projectile can cross height h twice in upward and downward flight respectively. So there are two points on the parabola a height h and with different horizontal distances.



The equation of trajectory is

$$y = x\tan\alpha - \frac{gx^2\sec^2\alpha}{2V_0^2}$$

putting $v_0 = \sqrt{2gH}$

$$y = x\tan\alpha - \frac{gx^2\sec^2\alpha}{2 \cdot 2gH}$$

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$$y = x\tan\alpha - \frac{g x^2 \sec^2\alpha}{4gH}$$

when $y = h$

Then

$$h = x\tan\alpha - \frac{x^2\sec^2\alpha}{4H}$$

$$\Rightarrow 4hH = (4H\tan\alpha)x - x^2\sec^2\alpha$$

$$x^2\sec^2\alpha - (4H\tan\alpha)x + 4hH = 0 \rightarrow (1)$$

This is quadratic in x and has two roots in x:

Let $x_1 = OA$, $x_2 = OB$.

From (1)

$$\text{Sum of roots} = x_1 + x_2 = \frac{4H\tan\alpha}{\sec^2\alpha}$$

$$= 4H\sin\alpha\cos\alpha$$

$$\text{Product of roots} = x_1x_2 = \frac{4hH}{\sec^2\alpha}$$

$$= 4hH\cos^2\alpha$$

$$AB = x_2 - x_1 = \sqrt{(x_1+x_2)^2 - 4x_1x_2}$$

$$= \sqrt{16H^2\sin^2\alpha\cos^2\alpha - 16hH\cos^2\alpha}$$

$$= 4\cos\alpha\sqrt{H(H\sin^2\alpha - h)}$$

Now if the shell has covered horizontal equal to AB before elevation from point O, then shell may cross the wall AP. Thus the gun can be fired within the strip OO — AB of plane so that the shell may cross the wall AP.

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Thus length of the strip $= 4\cos\alpha\sqrt{H(H\sin^2\alpha - h)}$

2nd Condition

Now the length of strip OO' must have some real value i.e

$$H\sin^2\alpha - h > 0$$

$$\Rightarrow H\sin^2\alpha > h$$

or

$$h < H\sin^2\alpha$$

Note # Since if $H\sin^2\alpha - h = 0$; then $OO' = AB = 0$; then the point of projection cannot move and we have.

$$H\sin^2\alpha = h$$

Maximum height reached $= \frac{V_0^2\sin^2\alpha}{g}$

$$= \frac{2gH\sin^2\alpha}{g}$$

$$= H\sin^2\alpha = h$$

$\Rightarrow$ top of the wall touches the vertex of the trajectory

Q: 15 # A shell fired with speed V at an elevation θ hits an airship at height H, with which is moving horizontally away from the gun with speed v_0. Show that if

$$(2V\cos\theta - v_0)(V^2\sin^2\theta - 2gH)^{1/2} = v_0 V\sin\theta$$

then the shell might also have hit the airship if the latter had remained stationary in the position it occupied when the gun was actually fired.

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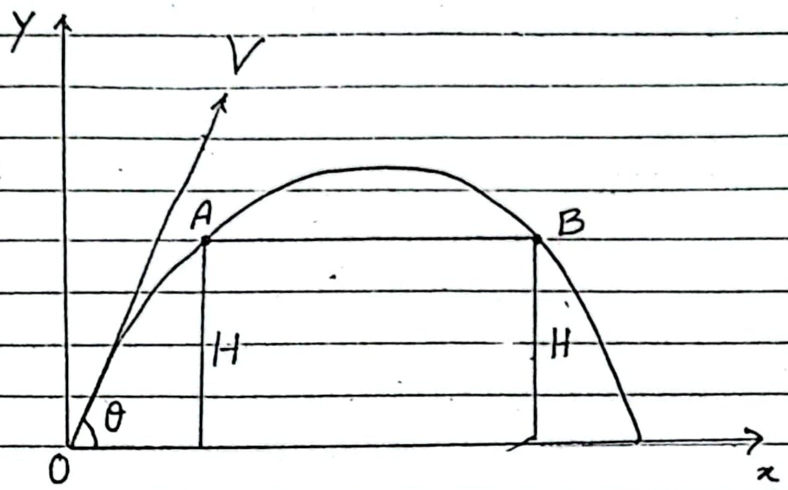
Sol #



Let the gun be at point O and airship be at point A initially. The shell would have hit the airship if the airship had been at rest provided A lies on the trajectory of the shell, at height H. So, when the shell is fired from the gun at O, the airship moves at constant height H, away from A. The shell has only the possibility to hit the airship at height H on the trajectory of the shell.

Let the shell hit the airship at point B after t seconds when it is fired.

Motion of the Shell

For shell t will be time from O to B at heigh H.

Equation of motion is

$$H = (V\sin\theta)t - \frac{1}{2}gt^2$$

$$\Rightarrow gt^2 - (2V\sin\theta)t + 2H = 0$$

$$t = \frac{2V\sin\theta \pm \sqrt{4V^2\sin^2\theta - 8gH}}{2g}$$

$$= \frac{V\sin\theta \pm \sqrt{V^2\sin^2\theta - 2gH}}{g}$$

These two values of time to cover height H

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are time for path from O to A and time for path from O to B. Let t_1 & t_2 be these times. Then

$$t_1 = \frac{V\sin\theta - \sqrt{V^2\sin^2\theta - 2gH}}{g} \quad \text{(Smaller time)}$$

$$t_2 = \frac{V\sin\theta + \sqrt{V^2\sin^2\theta - 2gH}}{g} \quad \text{(Greater time)}$$

Now the time taken by shell to cover horizontal distance $AB = t_2 - t_1$

$$= \frac{V\sin\theta + \sqrt{V^2\sin^2\theta - 2gH}}{g} - \frac{V\sin\theta - \sqrt{V^2\sin^2\theta - 2gH}}{g}$$

$$= \frac{2\sqrt{V^2\sin^2\theta - 2gH}}{g}$$

$\overline{AB}$ = (uniform horizontal vel) (time taken)

$$= V\cos\theta\,(t_2 - t_1)$$

$$\overline{AB} = \frac{2V\cos\theta\sqrt{V^2\sin^2\theta - 2gH}}{g} \longrightarrow (1)$$

Motion of the airship

Since the shell hit the ship at B, therefore the time taken by ship to move from A to B = time taken by the shell to reach at B from O

$$= \frac{V\sin\theta + \sqrt{V^2\sin^2\theta - 2gH}}{g} = t_2$$

So $\overline{AB} = v_0 t_2$

$$= \frac{v_0}{g}\left(V\sin\theta + \sqrt{V^2\sin^2\theta - 2gH}\right) \longrightarrow (2)$$

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∴ By ① and ②

$$\frac{2V\cos\theta}{g}\sqrt{V^2\sin^2\theta - 2gH} = \frac{v_0}{g}\left(V\sin\theta + \sqrt{V^2\sin^2\theta - 2gH}\right)$$

$$2V\cos\theta\sqrt{V^2\sin^2\theta - 2gH} = v_0 V\sin\theta + v_0\sqrt{V^2\sin^2\theta - 2gH}$$

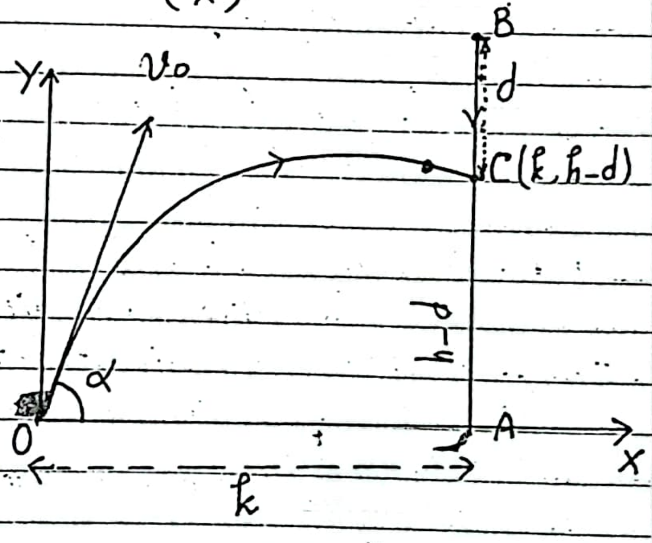
$$(2V\cos\theta - v_0)\sqrt{V^2\sin^2\theta - 2gH} = v_0 V\sin\theta$$

Q:16 # A ball is dropped from the top of a tower of height h. At the same moment, another ball is thrown from a point of ground at a distance k from the foot of tower so as to strike the 1st ball at a depth d. Show that the initial speed and the direction of projection of the 2nd ball are

$$\sqrt{\frac{g(h^2+k^2)}{2d}} \quad \text{and} \quad \tan^{-1}\left(\frac{h}{k}\right)$$

Sol #

Let AB = h be the tower and the 2nd ball be projected from point O with vel v_0 and angle of projection α. Suppose the balls strike at point C after time t. Taking O as origin, the co-ordinates of C are $(k, h-d)$.



Motion of the Dropped Ball

The first ball

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Covers vertical distance (d) in time t using

$$s = ut + \frac{1}{2}gt^2$$

$$d = 0.t + \frac{1}{2}gt^2$$

$$\Rightarrow d = \frac{1}{2}gt^2$$

$$\Rightarrow t = \sqrt{\frac{2d}{g}} \longrightarrow (1)$$

Motion of the Projected Ball

The parametric equations of trajectory are

$$x = v_0 \cos\alpha . t \longrightarrow (2)$$

$$y = v_0 \sin\alpha t - \frac{1}{2}gt^2 \longrightarrow (3)$$

Since point C(k, h-d) lies on the trajectory, therefore its co-ordinates will satisfy (2) & (3) and we have

$$k = v_0 \cos\alpha . t \longrightarrow (4)$$

$$h-d = v_0 \sin\alpha . t - \frac{1}{2}gt^2 \longrightarrow (5)$$

putting $t = \sqrt{\frac{2d}{g}}$ from (1)

$$k = v_0 . \cos\alpha . \sqrt{\frac{2d}{g}} \longrightarrow$$

$$h-d = v_0 \sin\alpha . \sqrt{\frac{2d}{g}} - \frac{1}{2}g\left(\frac{2d}{g}\right)$$

$$h-\cancel{d} = v_0 \sin\alpha \sqrt{\frac{2d}{g}} - \cancel{d}$$

$$h = v_0 \sin\alpha \sqrt{\frac{2d}{g}} \longrightarrow (7)$$

For Velocity of Projection v_0

squaring and adding

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⑥ & ⑦

$$k^2 + h^2 = V_o^2(\sin^2\alpha + \cos^2\alpha)\cdot\frac{2d}{g}$$

$$V_o^2 = \frac{g(k^2+h^2)}{2d}$$

$$V_o = \sqrt{\frac{g(k^2+h^2)}{2d}}$$

Angle of Projection

Dividing ⑦ by ⑥

$$\frac{h}{k} = \tan\alpha$$

$$\Rightarrow \alpha = \tan^{-1}\left(\frac{h}{k}\right)$$

EXTRA PROBLEMS

Q:1 # Prove that if the time of flight of a bullet over a horizontal range R is T seconds, the inclination of the direction to the horizontal is

$$\tan^{-1}\left(\frac{gT^2}{2R}\right)$$

Sol# Let α and v_o be the angle of projection and velocity of the bullet, then time of flight and the horizontal range are given by

$$T = \frac{2V_o\sin\alpha}{g}$$

$$R = \frac{2V_o^2\sin\alpha\cos\alpha}{g}$$

So $$\frac{T^2}{R} = \frac{4V_o^2\sin^2\alpha}{g^2} \times \frac{g}{2V_o^2\sin\alpha\cos\alpha} = \frac{2\tan\alpha}{g}$$

$$\Rightarrow \tan\alpha = \frac{gT^2}{2R}$$

$$\alpha = \tan^{-1}\left(\frac{gT^2}{2R}\right)$$